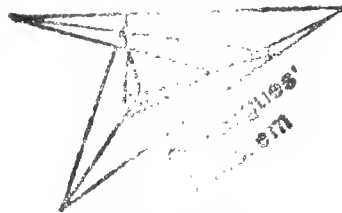


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UNIVERSITY OF CINCINNATI.

Bulletin No. 13.

Publications of the University of Cincinnati.

SERIES II.

Edited by HOWARD AYERS.

VOL. II.

*LECTURES ON THE THEORY OF
MAXIMA AND MINIMA OF FUNCTIONS
OF SEVERAL VARIABLES.*

(Weierstrass' Theory.)

By HARRIS HANCOCK, Ph. D. (Berlin), Dr. Sc. (Paris),
Professor of Mathematics.



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PREFACE.

IN his lectures at Berlin the late Professor Weierstrass often indicated the necessity of establishing fundamental parts of the *Calculus* upon a more exact foundation.

It has already been pointed out (*Annals of Mathematics*, Vols. IX., X., XI. and XII.) how the old rules and theories of the *Calculus of Variations* soon led to perplexities which appeared almost insurmountable. *Dirichlet's Principle* is found to have been established upon a weak structure, and we very soon find innumerable fallacies and difficulties when we seek to discuss in this manner *Minimal Surfaces* and the allied theory.

These difficulties may be overcome by subjecting the problems in question to a more rigorous treatment and by giving more emphasis to their analytic formulation.

In every Differential Calculus which I have seen [*cf. also Pierpont, Bull. of Amer. Math. Soc., July, 1898*] the Theory of Maxima and Minima is both inexact and inadequate, when several variables are treated. This subject, when made more rigorous, should evidently receive increased attention. Indeed, at the present state of mathematical science it seems that students should devote more attention to its study, for it has a high interest as a topic of pure analysis, and finds immediate application to almost every branch of mathematics. Further, the Theory of Maxima and Minima should receive more attention for its own sake—for example, in the solution of such problems as the determination of *the polygon which, with a given periphery and a given number of sides contains the greatest area*, the derivation of *the shortest line from a point to a surface, etc.* In the Calculus of Variations its use is really essential, while in Mechanics it may be shown that all problems which arise, may

be reduced to problems of Maxima and Minima; from it we may derive a proof of the *existence of the roots of algebraic equations* as also a method for *the reversion of series*.

I do not assume credit for the origin of any of the theories that are here set forth. In the presentation of the subject-matter I have followed Weierstrass' lectures delivered in the University at Berlin, my lectures being for the most part a reproduction of his lectures. In these lectures Weierstrass subjected to a more rigorous investigation the work which is in a great measure due to older writers, whom I have indicated in the context.

Before entering upon the 'Theory of Maxima and Minima' it seems advisable to give a short account of Weierstrass' Theory of Analytic Functions and to give more exact definitions of those functions to which the ordinary rules of differentiation are applicable. This investigation is carried out only so far as the present treatise seems to require. Certain theorems are also introduced upon which the later discussion depends. The 'Theory of Maxima and Minima' may be then presented in a clearer and more connected form.

My thanks are due to Messrs. Harry H. Steinmetz and Harold P. Murray of the University Press for their care in the printing of this work.

HARRIS HANCOCK.

McMICKEN HALL,
UNIVERSITY OF CINCINNATI.
JAN., 1903.

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$$[1] \quad y_{\lambda} = \sum_{\mu=1}^{\mu=n} (a_{\lambda\mu} + X_{\lambda\mu}) x_{\mu},$$

$$(\lambda=1, 2, \dots n)$$

where $a_{\lambda\mu}$ and $X_{\lambda\mu}$ are quantities which are subjected to certain conditions, then it is always possible to fix for the variables $x_1, x_2, \dots x_n$ and $y_1, y_2, \dots y_n$ definite limits $g_1, g_2, \dots g_n$ and $h_1, h_2, \dots h_n$ in such a manner that for every system of the y 's for which $|y_{\lambda}| < h_{\lambda}$ ($\lambda=1, 2, \dots n$) there exists one system of the x 's for which $|x_{\lambda}| < g_{\lambda}$ ($\lambda=1, 2, \dots n$) so that [1] is satisfied. (2) The solution of [1] has a similar form as the equations [1] themselves, viz:—

$$x_{\lambda} = \sum_{\mu=1}^{\mu=n} (b_{\lambda\mu} + Y_{\lambda\mu}) y_{\mu}.$$

$$(\lambda=1, 2, \dots n) \quad . \quad . \quad . \quad 103, 104$$

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CHAPTER I.

CERTAIN FUNDAMENTAL CONCEPTIONS IN THE THEORY OF ANALYTIC FUNCTIONS.

1. In the development of the conception of the analytic functions if we start with the simplest functions which may be expressed through arithmetical operations, we come first to the *rational functions of one or more variables*. The conception of these rational functions may be easily extended by substituting in their places *one-valued* functions, and first of all those which may be again expressed through arithmetical operations, viz.—sums of an infinite number of terms of which each is a rational function, or products of an infinite number of such functions.

The necessity at once arises of developing the conditions of convergence of infinite series and products, since such an arithmetical expression represents a definite function only for values of the variables for which it converges. Mere convergence, however, is not sufficient, if we wish to retain for the functions just mentioned the properties which belong to the rational and the ordinary transcendental functions. All such functions have derivatives, and in order to have this property the above expressions of one variable must converge *uniformly* (*gleichmässig*) in the neighborhood of each definite value.

2. When we say a function of one variable *converges uniformly*, we mean the following:*. It is assumed that the function in question has a definite value for $x = x_0$. We next consider all values of x for which $x - x_0$ does not exceed a definite quantity d . We shall then suppose that the series is to be convergent for a given value of x that lies within this interval. In order that this series converge uniformly, it must be possible, after we have

*A different definition is given by Weierstrass (Collected works, vol. II., p. 202 and Zur Functionenlehre, §1.)

assumed an arbitrarily positive quantity δ and when a remainder $R_n(x)$ has been separated from the series, to find a positive integer m so that

$$|R_n(x)| \leq \delta, \text{ where } n > m$$

for all values of x in this interval. [Cf. *Dini, Theorie der Functionen*, p. 137. Translated into German by Lüroth and Schepp.]

3. The conditions of uniform convergence being retained, it is essential that all the transcendental functions have a property in common: If we take a value x_0 within the *region of convergence* (Convergenzbezirk) in which these functions considered as functions of one variable converge uniformly, then they may be represented for all the values of x in the neighborhood of x_0 as series which proceed according to positive integral powers of $x - x_0$. For example,

$$f(x) = f(x - x_0 + x_0) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots,$$

where $a_0, a_1, a_2, \dots$ are definite functions of x_0 .

From this it follows that they may be differentiated and a number of other properties are immediate consequences.

4. Starting from the same standpoint as in the case of one-valued functions we may in a similar manner give to *many-valued* functions a definition which is far reaching in its generality. For this purpose the conception of the usual operations must be extended to several variables. It is then easy to extend in the required manner the conception of uniform convergence.

Let $a_1, a_2, \dots, a_n$ be a definite system of values of the variables $x_1, x_2, \dots, x_n$ within the region of uniform convergence, and consider only the values of $x_1, x_2, \dots, x_n$ for which $x_1 - a_1, x_2 - a_2, \dots, x_n - a_n$ do not exceed certain limits $d_1, d_2, \dots, d_n$. The function may be then represented through an ordinary series which proceeds according to integral powers of $x_1 - a_1, x_1 - a_2, x_2 - a_2, \dots, x_n - a_n$, and consequently may be differentiated; in short, it behaves—as Weierstrass was accustomed to express it—like an integral rational function in the neighborhood of a definite position within the interior of the region of uniform convergence.

5. The definition of arithmetical functions being established as in the preceding articles, the conception of *arithmetical de-*

dependence among several variables may be defined as follows: If we represent a function which has been formed as indicated above by $F(x_1, x_2, \dots x_n)$, then $F(x_1, x_2, \dots x_n) = 0$ expresses a certain dependence among the variables $x_1, x_2, \dots x_n$; that is, among the infinite number of systems of values for which the function has a meaning, those which satisfy this equation, are to be taken. There exists, therefore, among $x_1, x_2, \dots x_n$ a dependence of a similar character, as in the case of algebraic equations. If we choose the quantities $x_1, x_2, \dots x_n$ such that the equations $F_1 = 0, F_2 = 0, \dots F_m = 0$, where $m < n$, are true, then we have a dependence among the quantities $x_1, x_2, \dots x_n$ defined in such a way that at all events we can choose arbitrarily not more than $n - m$ of the variables, since the remaining m variables are determined.

6. In conformity with what was given in articles 4 and 5 we may develop also the conception of many valued functions.

Suppose, for example, a function of two variables x and y is given; then we may consider all the systems of values (x, y) in which x has a prescribed value. For such a value of x , several values of y may be found. We consider, then, y as a function of x , and this function will be a *many-valued* function.

The fact that by proceeding in this way we cannot grasp all the possible different forms of function has given occasion to present the conception of analytic functions in a different manner in that we make a characteristic property for the foundation of the definition.

Weierstrass wished in no wise to reject the mode of conception that has been used up to this point, in accordance with which we have to define every analytic dependence through quantities that may in reality be expressed in an arithmetical manner; for it will, in all probability, finally result that, however the analytic function may be defined, every dependence may be represented in the form indicated.

7. The following considerations will serve to explain the remarks of the previous article. Cases of dependence have been known for a long time where it could not be proved that they could be represented in an arithmetical manner. As an example let any analytic dependence exist between two variables and limit one of the variables x to a definite region; then the other varia-

ble y may be represented through x . We wish to express this representation in a form that remains continuously true. If to a value of the one variable a transcendental function of the other variable corresponds, we do not see the possibility of expressing one variable arithmetically in terms of the other variable.

Many examples are known in which the idea is conceived of expressing the variables x and y in terms of a third variable, and thereby representing both of the original variables as one-valued functions of a third variable in such a way that if we give to this variable all possible values we have all systems of values (x, y) . The simplest example is perhaps the following:

If we take the equation $z=x^y$, where x and y are to be two independent variables, then it is in no wise possible to express the above dependence in an arithmetic form; that is, in a form in which transcendental functions do not appear. But if we introduce a third variable t , and write $x=e^t$, then we have $x^y=e^{yt}$, and consequently $x=e^t$, $z=e^{yt}$.

Thus x and y are expressed in one-valued form by the help of t .

Much use has already been made of such dependence; for example, in the differential equation of the hypergeometric series.

This differential equation has among others two independent solutions which are transcendental functions of x . It was not at first known how to express this connection, but it was afterwards found that if we exclude from the values which the variable x can take those values which belong to a definite line (namely, the real values from $+1$ to $+\infty$), then for all remaining values the series may be expressed through a sum of rational functions.

Professor Poincaré has shown:* *If between x and y an algebraic equation exists, then all the systems of values may be expressed in the form indicated.* He next showed that the same could be effected for the linear differential equations by the help of functions, to which Fuchs had first called attention, and

* See Bulletin de la Société mathématique de France (t. XI, 1883).

See also lectures II and III, delivered in the Cambridge Colloquium, by Professor Osgood [Bulletin of the Amer. Math. Society, 1898].

which Poincaré calls "*Fuchsian Functions.*" If we have any linear differential equation, then we may express the two variables x, y as one-valued functions of a third variable.

The functions which have been introduced by Poincaré are extremely useful when we have to do with the linear differential equation of the second order. Poincaré found finally the following general theorem: *if in general there exists an analytic dependence between two variables x and y , it is always possible to represent x and y as one-valued functions of a third variable.* In this manner the study of many-valued functions is reduced to the study of one-valued functions. However this is not a simple task, for if we wish in reality to make this representation in the case of a linear differential equation, we encounter many technical difficulties. Nevertheless it is essential to prove that there exist such representations.

Further progress has not been made in this direction, but Weierstrass* thought that this end could be attained for the theory of functions, viz.: that it is always possible where an analytic dependence exists between two variables to express this dependence in a one-valued form which remains continuously true. In this treatment one may use, besides the variables, still other auxiliary variables.

8. After what has been given above it seems advisable to follow a different method than the one hitherto employed, so that we may come to a more comprehensive definition of the analytic functions, namely: a method indicated by a characteristic property of all the functions under consideration, *the representation through power-series for limited values of the variables.*

Suppose first an analytical relation exists between two variables; then this may be expressed through an analytic equation between x and y of the form†

$$P(x-x_0, y-y_0)=0,$$

if (x_0, y_0) is a definite *pair* of values of the variables. It is thus

*Weierstrass made this assertion in May, 1884.

See also *Problèmes Mathématiques* of Professor Hilbert in the *Comptes rendus* of the Congress of Mathematicians, Paris, 1900.

† The letter P has here, as throughout this discussion, simply the meaning "*power series.*"

seen that in the neighborhood of (x_0, y_0) there is an infinite number of systems of values which satisfy the equation. The collectivity of these pairs of values (x, y) is called an *analytic structure*—or *configuration* (Gebilde)—*in the realm* (Gebiet) *of the quantities* x, y .

As in the case of two variables, we shall proceed in a similar manner with several variables, among which an analytic connection exists that is expressible in arithmetical form. Let this connection be of such a nature that $m (< n)$ of the variables are in general determined through the remaining $n-m$. If, then, $(a_1, a_2, \dots a_n)$ represents a definite system of values of the variables, there exist m equations of the form

$$P(x_1-a_1, x_2-a_2, \dots x_n-a_n)=0,$$

which are to be satisfied for $x_1=a_1, x_2=a_2, \dots x_n=a_n$. In the neighborhood of the position $(a_1, a_2, \dots a_n)$ there are, then, an infinite number of other systems of values $(x_1, x_2, \dots x_n)$ which satisfy the same m equations. These define an *analytic structure in the realm of the quantities* $x_1, x_2, \dots x_n$. (Cf. Chap. IV, Art. 3.).

9. Analytical structures, as above defined, may be represented in a different manner. Since the equation between x and y begins with terms of the first dimension we may either, on the one hand express $y-y_0$ through $P(x-x_0)$ or $x-x_0$ through $P(y-y_0)$, or, on the other hand, if the coefficient of $x-x_0$, or $y-y_0$ is equal to zero, it is possible to express only $y-y_0$, or only $x-x_0$ as integral power-series of $x-x_0$ or $y-y_0$. In order that this distinction be not necessary, we take any function t which begins with terms of the first dimension in $x-x_0, y-y_0$ (see Art. 15); we may then always express the two quantities x, y as power-series of t . Through the assumption of such an equation it is always possible to include within certain limits all the systems of values $(x-x_0, y-y_0)$ which satisfy this equation. These values must *firstly*, satisfy the given equation, and *secondly*, afford all the systems of values which satisfy it.

These considerations may be extended at once to equations between several variables. If we have a certain number of equations between $x_1-a_1, x_2-a_2, \dots x_n-a_n$, and if we limit these equa-

tions to terms of the first dimension, we have linear homogeneous equations of the first dimension the number of which we assume to be $m < n$.

If we can express m of the quantities $x_1 - a_1, x_2 - a_2, \dots, x_n - a_n$ through the remaining $n - m$, it is always possible so to derive n power series of the $n - m$ quantities $t_1, t_2, \dots, t_{n-m}$ that they, substituted for $x_1, x_2, \dots, x_n$, *firstly*, satisfy the given equations, and *secondly* they, if we give to $t_1, t_2, \dots, t_{n-m}$ all possible values, afford all the systems of values $(x_1, x_2, \dots, x_n)$ which satisfy those equations, when certain limits are fixed for the absolute values of $x_1 - a_1, x_2 - a_2, \dots, x_n - a_n$, or also *secondly*, that with infinitely small values of the t 's they afford all the systems of values of the quantities $x_1, x_2, \dots, x_n$ which lie infinitely near the position $(a_1, a_2, \dots, a_n)$ [see again Art. 15].

If now, reciprocally, we assume n power-series of $t_1, t_2, \dots, t_{n-m}$ and write these equal to $x_1, x_2, \dots, x_n$, the collectivity of the systems of values $(x_1, x_2, \dots, x_n)$ offered through these equations constitute a *structure of the $(n - m)^{\text{th}}$ kind (Stufe) in the realm of the quantities $x_1, x_2, \dots, x_n$* .

Take n power-series $\Phi_1(t), \Phi_2(t), \dots, \Phi_n(t)$ and write $x_1 = \Phi_1(t), x_2 = \Phi_2(t), \dots, x_n = \Phi_n(t)$; then through these equations a *structure of the first kind in the realm of the n quantities x* is defined; in a similar manner a *structure of the second kind* is defined through the equations

$$x_1 = \Phi_1(t_1, t_2), x_2 = \Phi_2(t_1, t_2), \dots, x_n = \Phi_n(t_1, t_2), \text{ etc.}$$

10. The analytic structures that were characterized in articles 8 and 9 have a very important property, viz., that they may be *continued* (fortgesetzt).

The conception of *continuation* belongs to the most important conceptions of the more recent theory of functions. If instead of a we consider a position a_1 in the region of convergence of the series that defines the analytic function, then we may resolve this series that proceeds according to powers of $x - a$ into a series that proceeds according to powers of $x - a_1$. This series has, then, another region of convergence besides the one originally given; in general, the two regions of convergence will be partly separated, so that consequently the latter series has, at certain positions, a meaning at which the original series no longer had a meaning. They are identical for all values of x which lie in the neighbor-

hood of a_1 . On the other hand it is easy to show that, if a definite property belongs to one-power series (for example, if it satisfies a definite differential equation) this property belongs to all the power-series within the whole region of convergence. We are therefore compelled to look upon the second power-series as a *continuation* of the first.

In this way we may derive an indefinite number of power-series from the first one, and it is easy to see that if such a power-series possesses one or more definite properties, which may be expressed in the form of equations, the same properties belong also to each of the continuations.

It may be further shown that if a power-series $P(x-a_1)$ can be derived from another power-series $P(x-a)$, then this property of being derived is true also when performed backwards, either directly or also by the aid of other power-series. It is also clear that if we consider two power-series in $x-a'$, $x-a''$ which are derived from a power-series $x-a$, these series may be derived the one from the other. The collectivity of the power-series which may be derived from one single power-series as indicated, forms, in a certain measure, a whole which is closed in itself.

11. According to the previous article we shall define an analytic function of one variable as follows: *Consider a power-series of x assumed or given in any manner; let x' be a definite value of the variable x . Then three things may happen:*

(1) *x' may lie in the region of convergence of a series that is derived from the given series; the value for $x=x'$ of this series is a value of the analytic function which is determined through the original series. In other words: If with Weierstrass we call the original series and every series derived from this one with regard to the function which they represent, a FUNCTION-ELEMENT (Functionenelement), then the first possibility consists in that, if any FUNCTION-ELEMENT is given, the definite value x' lies in the region of convergence of a function-element which is derived from the given one.*

(2) *It may happen that x' does not lie in the region of convergence of any series that has been derived in this manner and that also we cannot derive from the original function-element another function-element whose region of convergence*

can come as near to the point x' as we wish. In this case the function does not exist for $x=x'$.

(3) Although we cannot find a power-series within which x' lies, nevertheless, it sometimes happens that we may still derive elements whose regions of convergence contain positions which can come as near to the point x' as we wish.* Whether we can then define the function for $x=x'$ by the consideration of boundary conditions must in each case be considered for itself.

If we have case (1) before us, then the function is defined not only for every value x' but also for all values in the neighborhood of x' and has for these values the character of an integral function.

12. The definition of an analytic function given in the preceding article is preferable to other definitions from the fact that the existence of general analytic functions is at once recognized; in short, that we have in our power, in our possession, all possible analytic functions. Every possible power-series within a region of convergence gives rise to the existence of a definite analytic function. Moreover, one must assume the duty of proving in the case of every example, that it leads to just such functions.

For this reason investigations are necessary of which in the olden times we find no trace. If we have a differential equation, we must begin with the proof that the functions which satisfy the differential equation arise from such function-elements that were explained in article 11; therefore, we must first show if y is the unknown function and x is the variable of the differential equation, that this equation can be satisfied through $y=P(x-a)$. Reciprocally, we must show that, if any variable quantity y is so connected with another variable quantity x that it satisfies the differential equation, then it may be derived from one single function-element in the manner indicated. This last proof is of especial importance in the application of analysis to geometrical mechanics.

When a problem is given in mechanics, we have to represent the coordinates of the moving point as functions of the time. Only real values are permitted in this representation. We can

* Cf. Chap. IV, Art. 3.

not therefore *à priori* know whether the required functions are analytic or not.

These functions are generally defined through differential equations. We shall give the simplest case as an example. Suppose we have a system of points that attract each other according to an analytic law, and let $x_1, x_2, \dots x_n$ be the coordinates of these points. If the motion is a free one, we have the differential equation in the form

$$\frac{d^2 x_1}{dt^2} + \dots = F(x_1, x_2, \dots x_n),$$

where F denotes a given function of $x_1, x_2, \dots x_n$. With such a problem we have to prove before everything else that the required functions of time are analytic functions. If for the point $t=t_0$ the initial position and the initial velocity are given, then in the neighborhood of the initial position we can find power-series, and we have to show that through these power-series the required functions are completely determined.

13. If we start with the definition of a function given in Art. 11, and if we have for a definite value x' a definite value of the function, then this value depends essentially upon the way and manner how we come to x' , that is, upon the choice of the values $a_1, a_2, \dots a_n$, by means of which we have come from one element to the derived element in question. From this it may be conjectured that one and the same function can have different values for the same values of the argument; these values can not be regarded as distinct from one another, but are to be considered as *values* of the function.

If a_1 lies within the region of convergence of the first series, a_2 in that of the second, $\dots a_n$ in that of the n^{th} series, then we have a definite power-series $P(x-a_n)$. If now we take instead of $a_1, a_2, \dots a_{n-1}$, $n-1$ other values $a'_1, a'_2, \dots a'_{n-1}$, then we can have another power-series $P_1(x-a_n)$, so that the function for every value x' in the region of convergence of this series has a different value than in the region $P(x-a_n)$.

Accordingly, we may offer the following definition: A function is called *one-valued* on the position a' , if in the neighborhood of a' we may derive a function-element from the original function-element and always have only *one* function-element along what-

ever path we have come from a to a' . If we have several function-elements, then the function is said to be *many-valued*; it may be *infinitely* many-valued.

Suppose next we have a *system of functions* of one variable x . Let n function-elements of x be given. Each of the elements determines an analytic function. It must be shown how the values of these n elements are arranged with respect to one another. If the functions are all or in part many-valued, the question arises how are we to arrange them with respect to their different values and with respect to one another. If, for example, we have two algebraic equations between the quantities x and y , we have through the elimination of y an equation for the determination of the possible values of x , and by analogy we may through the elimination of x form an equation for y . We cannot write down an arbitrary value of y for every x , but in general for each of the possible values of y we can take only one value of x .

This investigation is very simple in the case before us: We have values belonging in *common* to n functions that are defined in the given manner, if we always make use of the same intermediary a 's for the determination of the functions. In order to have a system of values belonging in common to the functions, for $x=x'$, we take a' in such a manner that it lies within the region of convergence of all the function-elements, the latter becoming $P(x-a')$. In the same way we take a quantity a'' which lies within the region of convergence of all the function-elements $P(x-a')$. These become $P(x-a'')$. Continuing in this manner we have a system of values belonging in *common* to the function.

In algebra and in all branches of mathematics, where the connection of functions is defined through equations, it must be shown that we have in reality by this procedure the values which belong in common to the functions.

These definitions may be extended to functions of several variables and to systems of functions of several variables.

We must yet show how functions of one variable are to be defined for the boundaries. This investigation will be reserved, however, for another occasion (Art. 19), where we shall present the conception of an analytic function in a different manner.

By means of these equations we may represent $x_1 - a_1, x_2 - a_2, \dots, x_m - a_m$ as power-series in the remaining $n - m$ differences, the formal procedure being as follows*:

We write $X_1' = 0, \dots, X_m' = 0$, and thus obtain for $x_1 - a_1, \dots, x_m - a_m$ expressions which represent the first approximations. These are substituted in $X_1', \dots, X_m'$ and the resulting expressions are reduced so as to contain only terms of the second dimension. Continuing this process, we may represent the required expressions to any degree of exactness desired.

We accomplish the same in the following manner: We write for m of the quantities $x_1 - a_1, \dots, x_n - a_n$ power-series with indeterminate coefficients; and it is seen that these coefficients may be uniquely determined.

It is a fundamental theorem in the theory of functions that these power-series are convergent as soon as the differences $x - a$ which enter into them do not exceed certain limits, and further, that these power-series satisfy the given equations. (See Chap. IV, Art. 25 *et seq.*)

15. The problem of the preceding article may be solved in the following more symmetric manner, in which none of the variables is given preference over the others.

Besides the equations given above, we introduce others which are likewise expressed in power-series:

$$\begin{aligned} \text{Let } c_{m+1,1}(x_1 - a_1) + \dots + c_{m+1,n}(x_n - a_n) + X_{m+1} &= t_1, \\ c_{n,1}(x_1 - a_1) + \dots + c_{n,n}(x_n - a_n) + X_n &= t_{n-m}. \end{aligned}$$

The quantities c are arbitrarily chosen, in such a manner, however, that the determinant,

$$\begin{vmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,m} & c_{1,m+1} & \dots & c_{1,n} \\ c_{2,1} & c_{2,2} & \dots & c_{2,m} & c_{2,m+1} & \dots & c_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{m,1} & c_{m,2} & \dots & c_{m,m} & c_{m,m+1} & \dots & c_{m,n} \\ c_{m+1,1} & c_{m+1,2} & \dots & c_{m+1,m} & c_{m+1,m+1} & \dots & c_{m+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n,1} & c_{n,2} & \dots & c_{n,m} & c_{n,m+1} & \dots & c_{n,n} \end{vmatrix} \begin{matrix} > \\ < \end{matrix} 0.$$

* See also Chap. III, Arts. 3-6; Chap. IV, Arts. 25 and 26.

Proceeding in a similar manner as in Art. 14, we write the quantities $\sum$ equal to zero, and we thus have a system of n linear equations through which we can express the n differences $x_1 - a_1, \dots, x_n - a_n$ through $t_1, t_2, \dots, t_{n-m}$:

$$x_v - a_v = e_{v,1} t_1 + e_{v,2} t_2 + \dots + e_{v,n-m} t_{n-m} + \sum'_v.$$

($v = 1, 2, \dots, n$).

With the help of these equations we can express $x_1 - a_1, \dots, x_n - a_n$ as power-series in $t_1, \dots, t_{n-m}$.

To do this we again write $\sum'_v = 0$, and have only terms of the first dimension. We write the first approximations that have been thus obtained in $\sum'_v$ and by retaining the terms of the second dimension derive the second approximations, etc.

It may be proved that these power-series in t *formally* satisfy the given equation; that they possess a certain common region of convergence if we give certain fixed limits to $|t_1|, |t_2|, \dots, |t_{n-m}|$, that they consequently *in reality* satisfy the equations; and finally, that all the systems of values $(x_1, x_2, \dots, x_n)$ which lie in the neighborhood of $(a_1, a_2, \dots, a_n)$ and which satisfy the proposed equations are had in this way.

In the theory of maxima and minima we shall give strenuous proofs of the statements just made, and in this connection a theorem will be proved which is very important in the theory of the Calculus of Variations. (See Chap. IV, Art. 25 *et seq.*)

16. We make the following application of the theorem given in Arts. 14 and 15. In accordance with this theorem, if between n quantities $x_1, x_2, \dots, x_n$ there exist m equations in the form of power-series, then the differences $x_1 - a_1, \dots, x_m - a_m$ may be expressed through power-series of the $n - m$ remaining variables. Weierstrass said: "*Through the m equations a structure of the $(n - m)^{th}$ kind in the realm of the n quantities $x_1, x_2, \dots, x_n$ is defined.*"

In virtue of the theorem of Art. 15 these structures may be expressed in manifold other ways.

If we introduce here, as in Art. 15, the quantities $t_1, t_2, \dots, t_{n-m}$, we have a symmetric representation of this structure, namely:

$$x_\nu - a_\nu = P_\nu(t_1, t_2, \dots, t_{n-m}) \quad (\text{for } \nu=1, 2, \dots, n).$$

It follows from the method of the derivation of these expressions that they satisfy the given equations.

If we wish to find all systems of values which satisfy the given equations and in which $|x_1 - a_1|, \dots, |x_n - a_n|$ are less than ρ , we may always assume ρ so small that the required systems are represented through the above formulæ.

A structure of the $(n-m)^{\text{th}}$ kind is thus defined in the realm of the n quantities $x_1, x_2, \dots, x_n$.

This theorem would be of little importance if the following was not true: the structure is *not* a closed one within itself, but a structure which may be continued over its boundaries; it is, as Weierstrass expressed it, only an *element* of a complete structure. The question arises, how are all the remaining elements derived from this one. We shall direct our attention in the following paragraphs to the discussion of this question.

17. Suppose, both for simplicity and clearness, that $n-m=1$; then, if we agree to represent $x_1, x_2, \dots, x_m$ as power-series of the last variable x_n , the functions $P(x_n - a_n)$ which have been so defined may be continued as in Art. 13. We need only assume a position a'_n in the region of convergence $P(x_n - a_n)$ and transform this series into $P_1(x_n - a'_n)$. We thus have in general a continuation as soon as the realm of convergence of the second series extends outside of the original realm.

If we make use of the quantity t , the continuation may be expressed in a much more general and symmetric form. As this quantity t is in general arbitrary, it is possible to express $x_1, x_2, \dots, x_n$ through t in many ways. Under the assumption that $n-m=1$, we shall limit this investigation to a structure of the form,

$$x_\nu - a_\nu = P_\nu(t) \quad (\nu=1, 2, \dots, n).$$

We are thus freed from an assumption tacitly made that the series $P_\nu(t)$ begin with terms of the first dimension; for we can choose $P_\nu(t)$ quite arbitrarily, the only condition being that it must vanish for $t=0$.

Let us take instead of the variable t the auxiliary variable τ . There clearly exists an equation between t and τ , since, if in the

formula $P(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$, through which τ is defined, we write

$$x_1 - a_1 = P_1(t), x_2 - a_2 = P_2(t), \dots, x_n - a_n = P_n(t),$$

then we have τ expressed as $P(t)$. Reciprocally we may represent t as a power-series in τ .

Suppose that $t = a_1 \tau + a_2 \tau^2 + \dots$, where the a 's must satisfy the condition that the series converge for certain values of τ .

We recognize from this that the same element which is represented through

$$x_v - a_v = P_v(t)$$

may also be represented through

$$x_v - a_v = P_v(\tau) \quad (v=1, 2, \dots, n).$$

From these equations it is clear that to every value of τ there belongs a value of t , and we may assume τ so small that the corresponding value of t lies within the region of convergence of the original series $P_v(t)$. We see that both systems of formulæ represent the same systems of values.

On the other hand, if $a_1 > 0$, we may express τ as $P(t)$:

$$\tau = \beta_1 t + \beta_2 t^2 + \dots$$

If we substitute this expression in the second system of formulæ $x_v - a_v = P_v(\tau)$, there must again appear on the right-hand side $P_v(t)$. We may now choose τ and t so small that the power-series, which represent them, converge; hence to every value of τ there corresponds a value t and to every value of t there corresponds a value τ . Consequently the systems of values $(x_1, x_2, \dots, x_n)$ which belong to pairs of corresponding values t, τ are identical.

The structure may be expressed through the one or the other system of formulæ. We shall say briefly that we have transformed the *expressions* of the structure—not the structure.

We may accordingly define the continuation of such functions as follows:

Consider a system of formulæ having the form

$$x_v = \Phi_v(t) \quad (v=1, 2, \dots, n),$$

where the Φ 's are functions of t which may be expressed in the form of power-series.

In this manner a structure is defined, if to t all values are given for which $\Phi_1, \dots, \Phi_n$ converge.

A structure is defined through

$$x_\nu = \Psi_\nu(\tau) \quad (\nu=1, 2, \dots, n).$$

These two structures, which in general have nothing in common, may have a common position; they will then agree for a definite value t_0 of t and a definite value τ_0 of τ , so that

$$\Phi_\nu(t_0) = \Psi_\nu(\tau_0) \quad (\nu=1, 2, \dots, n).$$

However, there is nothing of especial interest in this. Yet it may happen that in the neighborhood of these positions the two structures completely *coincide*; that is, there corresponds to every value of t in a certain neighborhood of t_0 a value of τ in a certain neighborhood of τ_0 , so that we have continuously

$$\Phi_\nu(t) = \Psi_\nu(\tau) \quad (\nu=1, 2, \dots, n).$$

This is expressed analytically, if we write

$$t = t_0 + s, \quad \tau = \tau_0 + \sigma,$$

where s and σ are two new variables.

In order that the two structures correspond in the neighborhood of the two positions under consideration, Φ_ν and Ψ_ν must so correspond to each other that Φ_ν goes into Ψ_ν , if we write

$$s = a_1 \sigma + a_2 \sigma^2 + \dots,$$

and *vice versa*, Ψ_ν must become Φ_ν if we write σ equal to a power-series in s . Consequently, to every infinitely small value of s (or σ), which is smaller than ρ , where ρ is a fixed limit taken sufficiently small, there must correspond an infinitely small value of σ or (s), so that we have the equation

$$\Phi_\nu(t_0 + s) = \Psi_\nu(\tau_0 + \sigma).$$

In a similar manner as was seen above in the case of t and τ , there is here an equation between s and σ . This equation must be of the form just given, and we must further have $a_1 \neq 0$, if the additional condition is made that to every infinitely small value of

σ there is to correspond only one infinitely small value of s . Reciprocally, if a_1 is different from zero, then to every value of σ there is to correspond only one value of s , since in the development of σ in a power-series of s only integral powers can appear. Accordingly, if the two structures have a common position and besides are to coincide in the neighborhood of this position, it must be possible to derive the expressions of these structures from each other in the manner indicated.

If this is the case, then naturally the two structures fall partly together. The remaining part of the second structure is to be considered as a continuation of the first structure.

That this is conformable with our purpose is seen from the fact that each of the structures is expressible through the same analytic function; we must, therefore, regard as belonging in *common* to one another all structures that are represented through the same analytic function.

18. After the preceding article we can make clear the meaning of the complete structure that was defined in Art. 16, in the following manner:

We start from an elementary original structure G_1 . We then consider a second element G_2 which overlaps with G_1 in the neighborhood of any position. Then G_2 is a direct continuation of G_1 . In the same way let G_3 be a direct continuation of G_2 ; then G_3 is an indirect continuation of G_1 , etc. The collectivity of the elementary structures G form a complete structure closed in itself.

It makes no difference from which element we start. The collectivity of the systems of values $(x_1, x_2, \dots, x_n)$ to which we come in this manner, is a complete structure.

Weierstrass calls such a complete structure in the realm of the quantities $x_1, x_2, \dots, x_n$ a *monogenic structure*,* because all the elementary structures of the realm arise from one and the same source.

In accordance with this a curve of the second degree is a *monogenic* structure of the first kind in the realm (x, y) ; however, the system of two straight lines is a *non-monogenic* structure.

* In connection with the definitions of the structures given here, which are adequate for the present discussion, see a paper by Prof. Osgood (Bull. of American Math. Soc., June, 1898).

If we have a monogenic structure of the first kind in the realm of n quantities, and if we seek all those positions in which one of the variables x has a definite value, then we may define all the appertaining systems of values of the remaining variables as a system of functions of x .

If we take a realm of two variables x, y and seek all the positions for which x takes a definite value x' , the corresponding values of y are called values of one and the same analytic function of x for $x=x'$.

In this structure y is therefore a function of x , and the function has for every given value of x as many values as there are positions on which x has that value.

19. We have now to show that the definition of Article 18 agrees with that given of an analytic function in Arts. 11 and 13, agrees in the sense: that all the systems of values of y which belong to x' are also to be had in the manner given in Arts. 11 and 13, with the distinction that here we have still other values of y which cannot be included in the previous definition.

A difficulty that attended the last definition may be avoided: We start from the representation $y=P(x-a)$ and derive from it all the possible power-series. We have thus defined a function of x . This structure has, however, boundary positions, and we have yet to determine whether the boundary values are to be counted with the structure.

If we make the explanation of the structure as in Art. 18, then this difficulty is avoided.

We consider only those pairs of values to which we can in reality come, as belonging to the structure which is defined through the given function of x, y .

Suppose that we again have the two equations

$$x=P_1(t), y=P_2(t) \dots \dots \dots (1)$$

For $t=t_0$, let $x=a, y=b$. The developments are therefore:

$$\begin{aligned} x-a &= a_1(t-t_0) + a_2(t-t_0)^2 + \dots, \\ y-b &= \beta_1(t-t_0) + \beta_2(t-t_0)^2 + \dots \end{aligned}$$

If we choose $a_1 \geq 0$, we have from the first of these equations

$$t-t_0 = \bar{P}(x-a).$$

Now, write this expression in the second of the equations, and it follows that

$$y-b=P(x-a)\dots\dots\dots(2)$$

The power-series on the right-hand side vanishes for $x=a$. If we limit ourselves to the values (x, y) which lie within a certain neighborhood of the position (a, b) , the positions which are represented through the equations (1) are also represented through the equation (2).

For this to be true:

(1) x must be chosen so near a that $t-t_0$ may *in reality* be represented through a power-series in $x-a$;

(2) x must lie so near a that t_0 lies within the region of convergence of the power-series in t ;

(3) If we write the expression of $x-a=\bar{P}_1(t-t_0)$ in equation (2), the resulting series must be convergent.

These conditions may always be satisfied, and we may therefore say that the equation (2) represents the same system of values in the neighborhood of (a, b) as the two original equations (1).

Reciprocally, if we start with such a power-series as (2) and derive from it two others (1), then the structure which is thus formed, that is, the collectivity of the systems of values (x, y) is one, whose definition is contained in the one just given. For by writing $x-a=t$, we have $y=P(t)$. We take in the region of convergence $P(x-a)$ the position a_1 and transform $P(x-a)$ into $P(x-a_1)$. If we then write $x-a_1=t'$, we have $t'=t+a-a_1$ or $t=t'+a_1-a$. The transformation, therefore, consists in the introduction of t' instead of t in the expression for y .

If, accordingly, we derive from $y=P(x)$ all the possible power-series and write each of them equal to y , the structure (x, y) which has been so defined is certainly contained in a monogenic structure.

This structure, in accordance with the definition of Art. 13, can contain yet certain other positions; for if we give to t_0 such a value that $a_1=0$, then $t-t_0$ cannot be written equal to $\bar{P}(x-a)$. Moreover, we have $t-t_0=P(\sqrt[\lambda]{x-a})$, if the power-series in $t-t_0$, through which $x-a$ is expressed, begins with $(t-t_0)^\lambda$, and therefore y is expressed in general through a power-series of a definite

root of $x-a$. These are the *singular* systems of values, the *boundary* positions, which we must count as belonging to the structure. For this reason the explanation given in Art. 18 of the structure is to be preferred to the one in Arts. 11, 13 and 16, since the first affords the boundary positions, and besides shows extraordinary symmetry.

20. The considerations given in the previous paragraphs for the case of two variables may be easily extended to any number of variables, say n .

Write

$$\begin{aligned} x-a &= a_1 (t-t_0) + a_2 (t-t_0)^2 + \dots, \\ y-b &= \beta_1 (t-t_0) + \beta_2 (t-t_0)^2 + \dots, \\ z-c &= \gamma_1 (t-t_0) + \gamma_2 (t-t_0)^2 + \dots, \\ &\dots\dots\dots \end{aligned}$$

where at least one of the quantities $a_1, \beta_1, \gamma_1, \dots$ is different from zero, say a ; then $t-t_0$ may be represented as a power-series in $x-a$, and then likewise $y-b, z-c, \dots$ as power-series in $x-a$.

If we continue these power-series by means of the same intermediary values, we have a system of $n-1$ functions of x . This system is contained in our structure. If, however, all the quantities $a_1, \beta_1, \dots$ were equal to zero, then we would have positions which could not be contained in the definitions given in Art. 13, and which must, moreover, be still counted as belonging to the structure.

We may accordingly define a system of analytic functions of one variable x in the following manner:

Let any monogenic structure of the first kind in the realm of the quantities $x_1, x_2, \dots, x_n$ be given; if we consider the positions at which x has one and the same value, then we have either only *one* such position, and then $x_1, x_2, \dots, x_n$ are *one-valued* functions of x , or there are several such values and then $x_1, x_2, \dots, x_n$ are *many-valued* functions of x .

We may also express this as follows: If we consider $x_1, x_2, \dots, x_n$ as functions of x , then we have all the systems of values $(x_1, x_2, \dots, x_n)$ that belong to a definite value of x , if we keep in sight all those positions in which x has the definite value under consideration.

21. At the close of this introduction to the General Theory of Functions we must yet consider systems of values of a structure which lie at infinity.

Weierstrass also made use of the sign $x-a$ for the case $a=\infty$ and understood by it nothing other than $\frac{1}{x}$, that is, a linear function which vanishes for $x=a=\infty$. We may therefore express $\frac{1}{x}$ through power-series as has been done with $x-a$ for finite values of a . Our definition is then true in general.

When several variables are present, we write

$$x_\nu - a_\nu = P_\nu(t) \quad (\nu=1, 2, \dots, n).$$

The power-series on the right vanishes for $t=0$, and x_ν goes into a_ν . If a_ν is infinite, this means $\frac{1}{x_\nu}$ may be expressed through such a power-series. If, further, these formulæ are in reality true (*i. e.*, if the series are convergent), then the position $(a_1, a_2, \dots, a_n)$ belongs to the structure. In this manner an element of the structure is defined which lies at infinity.

A position $(a_1, a_2, \dots, a_n)$ is always at infinity if only one of the quantities a is infinite.

CHAPTER II.

MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES
THAT ARE SUBJECTED TO NO SUBSIDIARY CONDITIONS.

1. *Introduction.*—Mathematicians have always been occupied with questions of maxima and minima. Already in Euclid's time one of the simplest problems of this character was: *Find the shortest line which may be drawn from a point to a line.* Methods for the solution of such problems were known long before the differential calculus, and certainly such problems exercised considerable influence upon the discovery of the calculus. Indeed, Lagrange wished to consider Fermat as the discoverer of the calculus, as he had been occupied with many problems of maxima and minima. He would in reality have probably come to the differential calculus had he started from a somewhat more general point of view. The theory of maxima and minima was rapidly developed along the lines of the calculus after its discovery. One was at first satisfied with finding the necessary conditions for the solution of a problem. These conditions, however, are seldom at the same time also sufficient. In order to decide this last point the discovery of further algebraic means was necessary.

2. We shall presuppose in the following discussion, unless it is expressly stated to the contrary, that not only the quantities that appear as arguments of the functions, but also the functions themselves are real, and that the functions, as soon as the variables are limited to a definite continuous region, have within this region everywhere the character of one-valued regular functions. We define *regular functions* in the following manner: *A function $f(x)$ is regular, if the function is defined for all values*

of x within certain limits of x , and if further for any value a of the argument within these limits, the development

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots$$

is possible; the series must be convergent and must in reality* represent the values of the function.

If we have a function of several variables $f(x_1, x_2, \dots, x_n)$ and if $a_1, a_2, \dots, a_n$ are any values of the variables, then the function is said to be *regular*, if the development

$$f(a_1+h_1, a_2+h_2, \dots, a_n+h_n)=P(h_1, h_2, \dots, h_n)$$

is possible; every term of the power-series is here a product of a power of h and a factor that is independent of h .

We limit ourselves to such functions that are analytic structures of the nature described in Arts. 8–21 of the previous chapter; only for such functions can we derive general theorems, since for other functions even the rules of the differential calculus are not applicable.

The problem of finding those values of the argument of a function $f(x)$, for which the function has a maximum or minimum value is not susceptible of a general solution, since there are functions which, in spite of the fact that they may be defined through a simple series or other algebraic expressions and which vary in a continuous manner, have an infinite number of maxima and minima within an interval which may be taken as small as we wish. Such functions do not come under the present investigation.

3. We say† that a function $f(x)$ of one variable has at a definite position $x=a$, a maximum or a minimum, if the value of the function for $x=a$ is respectively greater or less than it is for all other values of x which are situated in a neighborhood $|x-a|<\delta$ as near as we wish to a .

The analytical condition that $f(x)$ shall have for the position $x=a$,

a maximum, is expressed by $f(x)-f(a)<0$; } , for $|x-a|<\delta$.
 a minimum, is expressed by $f(x)-f(a)>0$; }

In the same way we say a function $f(x_1, x_2, \dots, x_n)$ of n variables

* And not only formally. (See Art. 15.)

† See Annals of Mathematics, Vol. IX, No. 6, p. 187.

has at a definite position $x_1=a_1, x_2=a_2, \dots x_n=a_n$ a maximum or a minimum, if the value of the function for $x_1=a_1, x_2=a_2, \dots x_n=a_n$ is respectively greater or less than it is for all other systems of values situated in a neighborhood

$$|x_\lambda - a_\lambda| < \delta_\lambda \quad (\lambda=1, 2, \dots n)$$

as near as we wish to the first position; and the analytical condition that the function $f(x_1, x_2, \dots x_n)$ shall have at the position $x_1=a_1, x_2=a_2, \dots x_n=a_n$

a maximum, is : $f(x_1, x_2, \dots x_n) - f(a_1, a_2, \dots a_n) < 0$,

a minimum, is : $f(x_1, x_2, \dots x_n) - f(a_1, a_2, \dots a_n) > 0$,

for $|x_\lambda - a_\lambda| < \delta_\lambda \quad (\lambda=1, 2, \dots n)$.

4. The problem which we have to consider in the theory of maxima and minima is, then, to find those positions at which a maximum or minimum really enters.

Since this problem for functions of one variable is exhaustively treated in the elements of the differential calculus, we shall discuss it here very briefly.

If x_1, x_2 are two values of x situated sufficiently near each other within a given region, then the difference of the corresponding values of the function is expressible in the form :

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1 + \epsilon(x_2 - x_1)),$$

where ϵ denotes a quantity situated between 0 and 1; or, if x_1 is written $=x$ and $x_2 = x+h$:

$$[1] \quad f(x+h) - f(x) = h f'(x + \epsilon h).$$

From this theorem may be derived Taylor's theorem in the form,*

$$[2] \quad f(x+h) - f(x) = h f'(x) + \frac{1}{2!} h^2 f''(x) + \dots \\ + \frac{1}{(n-1)!} h^{n-1} f^{(n-1)}(x) + \frac{1}{n!} h^n f^{(n)}(x + \epsilon h).$$

In the two formulæ last written, instead of $x+h$, write x and write a in the place of x ; they then become

* See Jordan, *Cours D'Analyse*, T. I, §§ 249-250.

$$[1^a] \quad f(x) - f(a) = (x-a)f'(a + \epsilon(x-a))$$

and

$$[2^a] \quad f(x) - f(a) = \frac{x-a}{1} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \\ + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(x + \epsilon(x-a)).$$

Since $f(x)$ is a regular, and consequently continuous, function, the same is true of all its derivatives. If $f'(a)$ is different from zero, then with small values of $h=x-a$, the value of $f(a+\epsilon h)$ is different from zero and has the same sign as $f'(a)$.

According to the choice of h , which is arbitrary, the difference $f(x)-f(a)$ can be made to have one sign or the opposite sign, if $f'(a)$ is either a positive quantity, or if it is a negative quantity. Hence the function $f(x)$ can have neither a maximum nor a minimum value at the position $x=a$, if $f'(a) \neq 0$.

We therefore have the theorem: *Maxima and minima of the function $f(x)$ can only enter for those values of x for which $f'(x)$ vanishes.*

It may happen that for the roots of the equation $f'(x)=0$, some of the following derivatives also vanish. If the n^{th} derivative is the first one that does not vanish for the root $x=a$, then from equation $[2^a]$ we have the formula:

$$f(x) - f(a) = \frac{(x-a)^n}{n!} f^{(n)}(x + \epsilon(x-a)),$$

and with small values of $h=x-a$, owing to the continuity of $f^{(n)}(x)$ the quantity $f^{(n)}(a+\epsilon h)$ will likewise be different from zero and will have the same sign as $f^{(n)}(a)$. If, therefore, n is an odd integer, we may always bring it about, according as h is taken positive or negative, that the difference $f(x)-f(a)$ with every value of $f^{(n)}(a)$ has either one sign or the opposite sign; consequently the function $f(x)$ will have at the position $x=a$, neither a maximum nor a minimum value.

If, however, n is an even integer, then h^n is always positive, whatever the choice of h may have been; consequently the difference $f(x)-f(a)$ is positive or negative according as $f^{(n)}(a)$ is positive or negative.

In the first case the function $f(x)$ has a minimum value at the position $x=a$, in the latter case a maximum.

Taking this into consideration, we have the following theorem for functions of one variable:

Maxima and minima of the function $f(x)$ can only enter for the roots of the equation $f'(x)=0$.

If a is a root of this equation, then at the position $x=a$ there is neither a maximum nor a minimum, if the first of the derivatives that does not vanish for this value is of an odd degree; if, however, the degree is even, then the function has a maximum value for the position $x=a$, if the derivative for $x=a$ is negative; a minimum, if it is positive.

5. In order to derive the analog for functions of several variables, we start again with Taylor's theorem for such functions. We may derive this theorem by first writing in $f(x_1, x_2, \dots, x_n)$

$$x_\lambda = a_\lambda + u(x_\lambda - a_\lambda), \quad (\lambda=1, 2, \dots, n)$$

where u is a quantity that varies between 0 and 1; we then apply to the function

$$\phi(u) = f(a_1 + u(x_1 - a_1), a_2 + u(x_2 - a_2), \dots, a_n + u(x_n - a_n))$$

Maclaurin's theorem for functions of one variable, viz.:

$$\begin{aligned} [3] \quad \phi(u) = & \phi(0) + \frac{u}{1!} \phi'(0) + \frac{u^2}{2!} \phi''(0) + \dots \\ & + \frac{u^{m-1}}{(m-1)!} \phi^{(m-1)}(0) + \frac{u^m}{m!} \phi^{(m)}(\epsilon u), \end{aligned}$$

and in this expression write $u=1$, as follows:

For brevity, denote by $f_k(x_1, x_2, \dots, x_n)$ the first derivative of $f(x_1, x_2, \dots, x_n)$ with respect to x_k , by $f_{k_1, k_2}(x_1, x_2, \dots, x_n)$ the derivative of $f(x_1, x_2, \dots, x_n)$ with respect to x_{k_1} and x_{k_2} , i. e.,

$$f_{k_1, k_2}(x_1, x_2, \dots, x_n) = \frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_{k_1} \partial x_{k_2}}, \text{ etc.}$$

We have, then,

$$\begin{aligned} \phi'(u) = & \sum_{k=1}^{k=n} \left\{ f_k(a_1 + u(x_1 - a_1), \dots, a_n + u(x_n - a_n)) (x_k - a_k) \right\}, \\ \phi''(u) = & \sum_{k_1, k_2} \left\{ f_{k_1, k_2}(a_1 + u(x_1 - a_1), \dots, \right. \\ & \left. a_n + u(x_n - a_n)) (x_{k_1} - a_{k_1}) (x_{k_2} - a_{k_2}) \right\}; \\ & \dots \dots \dots \end{aligned}$$

$$\phi^{(m-1)}(u) = \sum_{k_1, k_2, \dots, k_{m-1}} \left\{ f_{k_1, k_2, \dots, k_{m-1}}(a_1 + u(x_1 - a_1), \dots, a_n + u(x_n - a_n)) (x_{k_1} - a_{k_1}) \dots (x_{k_{m-1}} - a_{k_{m-1}}) \right\},$$

$$\phi^{(m)}(\epsilon u) = \sum_{k_1, k_2, \dots, k_m} \left\{ f_{k_1, k_2, \dots, k_m}(a_1 + \epsilon u(x_1 - a_1), \dots, a_n + \epsilon u(x_n - a_n)) (x_{k_1} - a_{k_1}) \dots (x_{k_m} - a_{k_m}) \right\}.$$

Hence, from [3] we have

$$\begin{aligned} \phi(u) - f'(a_1, a_2, \dots, a_n) &= \frac{u}{1} \sum_k \left\{ f_k(a_1, a_2, \dots, a_n) (x_k - a_k) \right\} \\ &\quad + \frac{u^2}{2!} \sum_{k_1, k_2} \left\{ f_{k_1, k_2}(a_1, a_2, \dots, a_n) (x_{k_1} - a_{k_1}) (x_{k_2} - a_{k_2}) \right\} \\ &\quad + \dots \\ &\quad + \frac{u^{m-1}}{(m-1)!} \sum_{k_1, k_2, \dots, k_{m-1}} \left\{ f_{k_1, k_2, \dots, k_{m-1}}(a_1, a_2, \dots, a_n) (x_{k_1} - a_{k_1}) \dots (x_{k_{m-1}} - a_{k_{m-1}}) \right\} \\ &\quad + \frac{u^m}{m!} \sum_{k_1, k_2, \dots, k_m} \left\{ f_{k_1, k_2, \dots, k_m}(a_1 + \epsilon u(x_1 - a_1), \dots, a_n + \epsilon u(x_n - a_n)) (x_{k_1} - a_{k_1}) \dots (x_{k_m} - a_{k_m}) \right\} \end{aligned}$$

From this it follows, if we write $u=1$:

$$\begin{aligned} f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n) &= \sum_k \left\{ f_k(a_1, a_2, \dots, a_n) (x_k - a_k) \right\} \\ &\quad + \frac{1}{2!} \sum_{k_1, k_2} \left\{ f_{k_1, k_2}(a_1, a_2, \dots, a_n) (x_{k_1} - a_{k_1}) (x_{k_2} - a_{k_2}) \right\} \\ &\quad + \dots \\ &\quad + \frac{1}{(m-1)!} \sum_{k_1, k_2, \dots, k_{m-1}} \left\{ f_{k_1, k_2, \dots, k_{m-1}}(a_1, a_2, \dots, a_n) (x_{k_1} - a_{k_1}) \dots (x_{k_{m-1}} - a_{k_{m-1}}) \right\} \\ &\quad + \frac{1}{m!} \sum_{k_1, k_2, \dots, k_m} \left\{ f_{k_1, k_2, \dots, k_m}(a_1 + \epsilon(x_1 - a_1), a_2 + \epsilon(x_2 - a_2), \dots, a_n + \epsilon(x_n - a_n)) (x_{k_1} - a_{k_1}) \dots (x_{k_m} - a_{k_m}) \right\}. \end{aligned}$$

6. We are not accustomed to Taylor's theorem in the form just given; in order to derive this theorem as it is usually given, we observe that upon performing the indicated summations, each of the indices $k_1, k_2, \dots$, independently the one from the other, takes all values from 1 to n , so that the λ^{th} term in the development is a homogeneous function of the λ^{th} degree in $x_1 - a_1, x_2 - a_2, \dots, x_n - a_n$. The general term of this homogeneous function may be written in the form

$$\frac{1}{\lambda!} D. N. (x_1 - a_1)^{\lambda_1} (x_2 - a_2)^{\lambda_2} \dots (x_n - a_n)^{\lambda_n},$$

$$\text{where } \lambda_1 + \lambda_2 + \dots + \lambda_n = \lambda,$$

$$D \text{ is the definite differential quotient } \left(\frac{\partial^\lambda f(x_1, x_2, \dots, x_n)}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2} \dots \partial x_n^{\lambda_n}} \right)_{a_1, a_2, \dots, a_n}$$

$$= f^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)}(a_1, a_2, \dots, a_n),$$

and N is the number of permutations of n elements of which $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively are alike,

$$i. e., N = \frac{\lambda!}{\lambda_1! \lambda_2! \dots \lambda_n!}.$$

Further, writing $x_k - a_k = h_k$, we have finally:

$$[4] \quad f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n)$$

$$= \sum_{k=1}^{k=n} \left\{ \left(\frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_k} \right)_{a_1, a_2, \dots, a_n} h_k \right\}$$

$$+ \frac{1}{2} \sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, a_2, \dots, a_n} h_\lambda h_\mu \right\}$$

$$+ \dots \dots \dots$$

$$+ \sum_{\lambda_1, \lambda_2, \dots, \lambda_n}^{\lambda_1 + \lambda_2 + \dots + \lambda_n = m-1} \left\{ f^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)}(a_1, a_2, \dots, a_n) \frac{h_1^{\lambda_1} h_2^{\lambda_2} \dots h_n^{\lambda_n}}{\lambda_1! \lambda_2! \dots \lambda_n!} \right\}$$

$$+ \sum_{\lambda_1, \lambda_2, \dots, \lambda_n}^{\lambda_1 + \lambda_2 + \dots + \lambda_n = m} \left\{ f^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)}(a_1 + \epsilon h_1, \dots, a_n + \epsilon h_n) \frac{h_1^{\lambda_1} h_2^{\lambda_2} \dots h_n^{\lambda_n}}{\lambda_1! \lambda_2! \dots \lambda_n!} \right\}$$

This is the usual form of Taylor's theorem for functions of several variables. In particular, when $m=1$, the above development is:

$$[5] \quad f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n) \\ = \sum_{k=1}^{k=n} \left\{ f_k(a_1 + \epsilon h_1, \dots, a_n + \epsilon h_n) h_k \right\}.$$

The function $f(x_1, \dots, x_n)$ is regular and continuous, as are consequently all its derivatives. If, therefore, the first derivatives of $f(x_1, x_2, \dots, x_n)$ are all, or in part, ≥ 0 for $x_1=a_1, \dots, x_n=a_n$, then they will also be different from zero for $x_1=a_1+\epsilon h_1, \dots, x_n=a_n+\epsilon h_n$, where the absolute values of $h_1, h_2, \dots, h_n$ have been taken sufficiently small; these derivatives will also be of the same sign as they were for $x_1=a_1, x_2=a_2, \dots, x_n=a_n$. If now we choose all the h 's zero, with the exception of one, which may be taken either positive or negative, we see that when the corresponding derivative has one sign or the opposite sign, we may always bring it about at pleasure that the difference

$$f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n)$$

is either a positive or a negative quantity, and consequently at the position $a_1, a_2, \dots, a_n$ neither a maximum nor a minimum value of the function is permissible.

We therefore have the following theorem:

Maxima and minima of the function $f(x_1, x_2, \dots, x_n)$ can only enter for those systems of values of $(x_1, x_2, \dots, x_n)$ which at the same time satisfy the n equations:

$$[6] \quad \frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \dots, \frac{\partial f}{\partial x_n} = 0.$$

It may happen that for the common roots of the system of equations [6] also still higher derivatives vanish. In this case we can in general only say that, if for a system of roots of the equations [6] all the derivatives of several of the next higher orders vanish, and if the first derivative which does not vanish for these values is of an odd order, the function, as may be shown by a similar method of reasoning as above, has certainly no maximum or minimum value.

7. If, however, this derivative is of an even order, then in the present state of the theory of forms of the n^{th} order in several variables there is no general criterion regarding the behavior of the function at the position in question. We therefore limit ourselves to the case where the derivatives of the second order of the function $f(x_1, x_2, \dots, x_n)$ do not all vanish for the system of real roots $a_1, a_2, \dots, a_n$ of the equations [6].

In this case we have a criterion in the formula,

$$[7] \quad f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n) \\ = \frac{1}{2} \sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1 + \epsilon h_1, \dots, a_n + \epsilon h_n} h_\lambda h_\mu \right\},$$

by which we may determine whether $f(x_1, x_2, \dots, x_n)$ has a maximum or a minimum value on the position $a_1, a_2, \dots, a_n$, since we may determine whether the integral homogeneous function of the second degree,

$$\sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1 + \epsilon h_1, \dots, a_n + \epsilon h_n} h_\lambda h_\mu \right\},$$

in the n variables $h_1, h_2, \dots, h_n$ is for arbitrary values of those variables continuously positive or continuously negative.

On account of their presupposed continuity, the quantities

$$\left(\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1 + \epsilon h_1, \dots, a_n + \epsilon h_n} \quad \text{and} \quad \left(\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, \dots, a_n}$$

with values of $h_1, h_2, \dots, h_n$ taken sufficiently small, differ from each other as little as we wish and are of the same sign; hence, with small values of the h 's the functions

$$\sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1 + \epsilon h_1, \dots, a_n + \epsilon h_n} h_\lambda h_\mu \right\} \\ \text{and} \quad \sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, \dots, a_n} h_\lambda h_\mu \right\}$$

have always the same sign, and we may therefore confine ourselves to the investigation of the latter function.

If we find that through a suitable choice of $h_1, h_2, \dots, h_n$ the expression

$$\sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, a_2, \dots, a_n} h_\lambda h_\mu \right\}$$

can be made at pleasure either positive or negative, the same will be the case with the difference $f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n)$, and consequently $f(x_1, x_2, \dots, x_n)$ has on the position $(a_1, a_2, \dots, a_n)$ neither a maximum nor a minimum value.

We therefore have as a second condition for the existence of a maximum or a minimum of the function $f(x_1, x_2, \dots, x_n)$ on the position $(a_1, a_2, \dots, a_n)$ that, in case the second derivatives of the function $f(x_1, x_2, \dots, x_n)$ do not all vanish at this position, the homogeneous quadratic form

$$\sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, a_2, \dots, a_n} h_\lambda h_\mu \right\}$$

must be always negative or always positive for arbitrary values of $h_1, h_2, \dots, h_n$.

THEORY OF THE HOMOGENEOUS QUADRATIC FORMS.

8. There are two kinds of integral homogeneous functions of the second degree, or, as they are usually called, *quadratic forms*, viz.:

I., *formæ indefinitæ*,* which with real values of the variables can become both positive and negative, and that, too, for values of the variables, whose absolute values do not exceed an arbitrarily small quantity;

II., *formæ definitæ*, which with real values of the variables have always the same sign.

We distinguish among the definite forms:

[1] Those which only vanish, when all the variables become zero, and

* Gauss, *Disq. Arithm.*, p. 271.

[2] Those which may also vanish for other values of the variables.

If our homogeneous function is an indefinite form, it is clear that the function $f(x_1, x_2, \dots, x_n)$ has neither a maximum nor a minimum upon the position $(a_1, a_2, \dots, a_n)$; for if the right-hand member of [7] is positive (say) for a definite system of values of the h 's, then in accordance with the definition of the indefinite quadratic forms we can find in the immediate neighborhood of the first system a second system of values of the h 's, for which the right-hand side of the equation [7] is negative; consequently, also, the difference

$$f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n)$$

is negative, so that therefore no maximum or minimum is permissible for the position $(a_1, a_2, \dots, a_n)$.

If, then, the second derivatives of the function $f(x_1, x_2, \dots, x_n)$ do not all vanish at the position $(a_1, a_2, \dots, a_n)$, it follows besides the equations [6] as a further condition for the existence of a maximum or a minimum of the function $f(x_1, x_2, \dots, x_n)$ that the terms of the second dimension in [4] must be a definite quadratic form, and, indeed, as will be shown later, one which can only vanish when all the variables become zero.

The question next arises: Under what conditions is in general a homogeneous quadratic form

$$[8] \quad \phi(x_1, x_2, \dots, x_n) = \sum_{\lambda, \mu} A_{\lambda, \mu} x_\lambda x_\mu$$

$$A_{\lambda, \mu} = A_{\mu, \lambda}$$

a *definite* quadratic form of the kind indicated?

9. Before we endeavor to answer this question, we must yet consider some known properties of the homogeneous functions of the second degree.

Suppose that in the function $\phi(x_1, x_2, \dots, x_n)$ in the place of $(x_1, x_2, \dots, x_n)$ homogeneous linear functions of these quantities

$$[9] \quad y_\lambda = \sum_{\mu=1}^{\mu=n} c_{\lambda, \mu} x_\mu \quad (\lambda=1, 2, \dots, n)$$

are substituted, which are subjected to the condition that inversely the x 's may be linearly expressed in terms of the y 's, and consequently the determinant

$$[10] \quad \begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{vmatrix} = \sum \pm c_{11} c_{22} \dots c_{nn} \geq 0.$$

The function $\phi(x_1, x_2, \dots, x_n)$ is thereby transformed into

$$[11] \quad \phi(x_1, x_2, \dots, x_n) = \psi(y_1, y_2, \dots, y_n).$$

Owing to this substitution it may happen that $\psi(y_1, y_2, \dots, y_n)$ does not contain one of the variables y , so that $\phi(x_1, x_2, \dots, x_n)$ is expressible as a function of less than n variables.

In order to find the condition for this, let us write

$$[12] \quad \phi_\lambda = \frac{1}{2} \frac{\partial \phi}{\partial x_\lambda} = \sum_{\mu} A_{\lambda\mu} x_\mu \quad (\lambda=1, 2, \dots, n).$$

If ψ is independent of one of the y 's, say y_n , so that consequently $\frac{\partial \psi}{\partial y_n} = 0$, then from the n equations

$$[13] \quad 2 \phi_\lambda = \sum_{\mu=1}^{\mu=n} \left(\frac{\partial \psi}{\partial y_\mu} \frac{\partial y_\mu}{\partial x_\lambda} \right) = \sum_{\mu=1}^{\mu=n} \left(c_{\mu\lambda} \frac{\partial \psi}{\partial y_\mu} \right) \quad (\lambda=1, 2, \dots, n)$$

we may eliminate the $m-1$ unknown quantities $\frac{\partial \psi}{\partial y_1}, \frac{\partial \psi}{\partial y_2}, \dots, \frac{\partial \psi}{\partial y_{n-1}}$. We thus have among the ϕ 's an equation of the form

$$[14] \quad \sum_{m=1}^{m=n} k_m \phi_m = 0,$$

where the k 's are constants.

Owing to equations [12] this means that the determinant of the given quadratic form vanishes, *i. e.*,

$$[15] \quad \sum \pm A_{11} A_{22} \dots A_{nn} = 0.$$

Reciprocally, it is easy to show that if the equation [15] is true, it is possible to express the function ϕ as a function of $n-1$ variables.

For we have

$$\begin{aligned}
 [16] \quad & \left\{ \sum_{\lambda} \left\{ \phi_{\lambda}(x_1, x_2, \dots, x_n) x'_{\lambda} \right\} \right\} = \sum_{\lambda, \mu} A_{\lambda \mu} x_{\mu} x'_{\lambda}, \\
 & \left\{ \sum_{\lambda} \left\{ \phi_{\lambda}(x'_1, x'_2, \dots, x'_n) x_{\lambda} \right\} \right\} = \sum_{\lambda, \mu} A_{\lambda \mu} x'_{\mu} x_{\lambda}, \\
 & = \sum A_{\mu \lambda} x'_{\lambda} x_{\mu} = \sum A_{\lambda \mu} x'_{\lambda} x_{\mu};
 \end{aligned}$$

and consequently

$$[17] \quad \sum \left\{ \phi_{\lambda}(x_1, x_2, \dots, x_n) x'_{\lambda} \right\} = \sum \left\{ \phi_{\lambda}(x'_1, x'_2, \dots, x'_n) x_{\lambda} \right\}.$$

There exists, further, the well-known Euler's theorem for homogeneous functions:

$$[18] \quad \sum_{\lambda} \left\{ \phi_{\lambda}(x_1, x_2, \dots, x_n) x_{\lambda} \right\} = \phi(x_1, x_2, \dots, x_n).$$

If now we assume that equation [15], or, what amounts to the same thing, an identical relation of the form [14] exists, and if we substitute in $\phi(x_1, x_2, \dots, x_n)$ the quantities $x_{\lambda} + t k_{\lambda}$ in the place of x_{λ} ($\lambda=1, 2, \dots, n$) and develop with respect to powers of t , we then have

$$\begin{aligned}
 \phi(x_1 + t k_1, x_2 + t k_2, \dots, x_n + t k_n) &= \phi(x_1, x_2, \dots, x_n) \\
 &+ 2 t \sum \left\{ k_{\lambda} \phi_{\lambda}(x_1, x_2, \dots, x_n) \right\} + t^2 \phi(k_1, k_2, \dots, k_n).
 \end{aligned}$$

It follows, when we take into consideration the equations [14] and [18], since the equation [14] is true for every system of values $(x_1, x_2, \dots, x_n)$, that

$$\phi(x_1 + t k_1, \dots, x_n + t k_n) = \phi(x_1, \dots, x_n).$$

Hence, if the equation [15] exists, or if the k 's satisfy the equation [14] for every system of values $(x_1, x_2, \dots, x_n)$, then $\phi(x_1, x_2, \dots, x_n)$ remains invariative, if $x_{\lambda} + t k_{\lambda}$ is written for x_{λ} , where t is an arbitrary quantity.

Consequently, it being presupposed that $k_v \leq 0$, if t is so determined that the argument $x_v + t k_v = 0$, we have

$$[19] \quad \phi(x_1, x_2, \dots, x_n) = \phi\left(x_1 - \frac{k_1}{k_v} x_v, x_2 - \frac{k_2}{k_v} x_v, \dots, \dots, x_{v-1} - \frac{k_{v-1}}{k_v} x_v, 0, x_{v+1} - \frac{k_{v+1}}{k_v} x_v, \dots, x_n - \frac{k_n}{k_v} x_v\right),$$

where ϕ is expressed as a function of less than n variables. We therefore have the theorem:

The vanishing of the determinant $\sum \pm A_{11} A_{22} \dots A_{nn}$ is the necessary and sufficient condition that a homogeneous quadratic function $\phi(x_1, x_2, \dots, x_n) = \sum_{\lambda, \mu} A_{\lambda\mu} x_\lambda x_\mu$ be expressible as a function of $n-1$ variables.

10. We return to the question proposed at the end of Art. 8; and in order to have a definite case before us, we shall assume that the problem is: *Determine the condition under which the function $\phi(x_1, x_2, \dots, x_n)$ is continuously positive.* The second case where $\phi(x_1, x_2, \dots, x_n)$ is to be continuously negative is had at once, if we write $-\phi$ in the place of ϕ .

We shall first show, following a method due to Weierstrass,* that every homogeneous function of the second degree $\phi(x_1, x_2, \dots, x_n)$ may be expressed as an aggregate of squares of linear functions of the variables.

11. In the proof of the above theorem we assume that $\phi(x_1, x_2, \dots, x_n)$ cannot be expressed as a function of $n-1$ variables; it follows, therefore, that the inequality

$$[20] \quad \sum \pm A_{11} A_{22} \dots A_{nn} \leq 0$$

is true, and it is not possible to determine constants k_i , so that the equation $\sum_{i=1}^{i=n} k_i \phi_i = 0$ exists identically.

* See also Lagrange, *Misc. Taur.*, I., p. 18, 1759; *Mecanique*, T. I., 3; Gauss, *Disq. Arithm.*, p. 271; *Theoria Comb. Observ.* p. 31, etc.

If, then, y represents a linear function of x having the form

$$[21] \quad y = c_1 x_1 + c_2 x_2 + \dots + c_n x_n,$$

and if g is a certain constant, then the expression $\phi - g y^2$, after the theorem proved above, can be expressed as a function of only $n-1$ variables, if the constants $k_1, k_2, \dots, k_n$ may be so determined that

$$\sum_{\lambda=1}^{\lambda=n} \left\{ k_{\lambda} (\phi_{\lambda} - g y c_{\lambda}) \right\} = 0,$$

or,

$$[22] \quad \sum_{\lambda=1}^{\lambda=n} k_{\lambda} \phi_{\lambda} - g y \sum_{\lambda=1}^{\lambda=n} k_{\lambda} c_{\lambda} = 0.$$

From the assumption made regarding [20], it follows on the one hand that the inequality

$$[23] \quad \sum_{\lambda} k_{\lambda} c_{\lambda} \geq 0$$

must exist. On the other hand, in virtue of the n linear equations,

$$[24] \quad \sum_{\mu} A_{\lambda \mu} x_{\mu} = \phi_{\lambda},$$

($\lambda = 1, 2, \dots, n$)

the quantities $x_1, x_2, \dots, x_n$ may be expressed as linear functions of $\phi_1, \phi_2, \dots, \phi_n$, and consequently by the substitution of these values of $x_1, x_2, \dots, x_n$ in [21] y takes the form

$$[25] \quad y = \sum_{\nu=1}^{\nu=n} e_{\nu} \phi_{\nu},$$

where the e_{ν} are constants, which are composed of the constants $A_{\mu \lambda}$ and c_{λ} .

But from equation [22] it follows that

$$y = \frac{\sum k_{\lambda} \phi_{\lambda}}{g \sum k_{\lambda} e_{\lambda}}.$$

Such a representation of the ϕ_λ , however, since we have to do with linear equations, can be effected only in one way.

$$\text{We therefore have } y = \frac{\sum_{\mu=1}^{\mu=n} k_\mu \phi_\mu}{g \sum_{\mu=1}^{\mu=n} k_\mu c_\mu} = \sum_{\lambda=1}^{\lambda=n} k_\lambda \phi_\lambda,$$

from which it follows that

$$k_\lambda = g e_\lambda \sum_{\mu=1}^{\mu=n} k_\mu c_\mu \quad (\lambda=1, 2, \dots, n).$$

Through the substitution of these values in [22] we have

$$\sum_{\lambda=1}^{\lambda=n} e_\lambda \phi_\lambda - g y \sum_{\lambda=1}^{\lambda=n} e_\lambda c_\lambda = 0;$$

consequently, owing to the relation [25], we have

$$[26] \quad \frac{1}{g} = \frac{\sum_{\lambda=1}^{\lambda=n} e_\lambda c_\lambda}{\sum_{\lambda=1}^{\lambda=n} e_\lambda c_\lambda} \text{ or } g = \frac{1}{\sum_{\lambda=1}^{\lambda=n} e_\lambda c_\lambda}.$$

This value of g may be expressed in a different form, for from [25] and [17] it follows that

$$y = \sum_{\nu=1}^{\nu=n} e_\nu \phi_\nu (x_1, x_2, \dots, x_n) = \sum_{\nu=1}^{\nu=n} x_\nu \phi_\nu (e_1, e_2, \dots, e_n).$$

Comparing this result with [21], we have

$$[27] \quad c_\nu = \phi_\nu (e_1, e_2, \dots, e_n) \quad (\nu=1, 2, \dots, n),$$

and consequently,

$$[28] \quad g = \frac{1}{\sum_{\lambda=1}^n e_{\lambda} \phi_{\lambda} (e_1, e_2, \dots, e_n)},$$

or, from [18]

$$g = \frac{1}{\phi (e_1, e_2, \dots, e_n)}.$$

Since the quantities $c_1, c_2, \dots, c_n$ are perfectly arbitrary except the one restriction expressed by the inequality [23], the quantities $e_1, e_2, \dots, e_n$ are in consequence of the equation [27] completely arbitrary with the one limitation resulting from [28], viz., the function ϕ cannot vanish for the system of values $(e_1, e_2, \dots, e_n)$: otherwise g would become infinite.

12. Reciprocally, if the quantities $e_1, e_2, \dots, e_n$ are arbitrarily chosen, but with the restriction just mentioned, and if g is determined through [28], it may be proved that the expression $\phi - g y^2$, where y has the form [25], may be expressed as a function of only $n-1$ variables.

For form the derivatives of this expression with respect to the different variables, and multiply each of the resulting quantities by the constants $e_1, e_2, \dots, e_n$. Adding these products and having regard to [28], we have

$$\sum e_{\lambda} \phi_{\lambda} - g y \sum e_{\lambda} \phi_{\lambda} (e_1, e_2, \dots, e_n) = \sum e_{\lambda} \phi_{\lambda} - y.$$

The expression on the right-hand side is zero from [25]. Hence we may choose n constants in such a way that the sum of the products of these constants and the derivatives of the expression $\phi - g y^2$ is identically zero. From this it follows by a similar method of reasoning as was given in connection with the equation [14] that the expression $\phi - g y^2$ may be expressed as a function of only $n-1$ variables.

13. If we represent $\phi (x_1, x_2, \dots, x_n) - g y^2$ by $\bar{\phi} (x_1, x_2, \dots, x_n)$, we may derive this function of $n-1$ variables from [19] by substituting $x_{\lambda} + t e_{\lambda}$ for x_{λ} ($\lambda=1, 2, \dots, n$) in $\bar{\phi}$; if one of these arguments is made equal to zero, we have as in art. 9

$$\phi(x_1, x_2, \dots, x_n) - g y^2 = \overline{\phi} \left(x_1 - \frac{e_1}{e_k} x_k, \dots, x_k - \frac{e_{k-1}}{e_k} x_k, 0, x_{k+1} - \frac{e_{k+1}}{e_k} x_k, \dots, x_n - \frac{e_n}{e_k} x_k \right),$$

or, if the new arguments are represented by $x'_1, x'_2, \dots, x'_{n-1}$,

$$\phi(x_1, x_2, \dots, x_n) - g x^2 = \overline{\phi}(x'_1, x'_2, \dots, x'_{n-1}).$$

Employing the same method of procedure with $\overline{\phi}(x'_1, x'_2, \dots, x'_{n-1})$ as was done with $\phi(x_1, x_2, \dots, x_n)$, we come finally to the function of only one variable, which being a homogeneous function of the second degree is itself a square. Hence we have the given homogeneous function $\phi(x_1, x_2, \dots, x_n)$ expressed as the sum of squares of linear homogeneous functions of the variables. If the coefficients of ϕ are real as also the quantities e , the coefficients g are also real and since the quantities e may with a single limitation be arbitrarily chosen, it follows that a transformation of such a kind that the result shall be a real one, may be performed in an infinite number of ways.*

14. If now the expression

$$[29] \quad \phi(x_1, x_2, \dots, x_n) = g_1 y_1^2 + g_2 y_2^2 + \dots + g_n y_n^2$$

is to be continuously positive for real values of the variables and equal to zero only when the variables themselves all vanish, then all the quantities $g_1, g_2, \dots, g_n$ must be positive; for if this was not the case, but g_1 (say) was negative, then since the y 's are independently of each other linear homogeneous functions of the x 's we could so choose the x 's that all the y 's except y_1 vanished and consequently contrary to our assumption $\phi(x_1, x_2, \dots, x_n)$ would be negative. Further none of the g 's can vanish; for if g_1 , say, was zero, we might so choose a system of values $x_1, x_2, \dots, x_n$ in which at least not all the quantities $x_1, x_2, \dots, x_n$ were zero that all the y 's vanished except y_1 , and consequently ϕ could then be zero without the vanishing of all the variables $x_1, x_2, \dots, x_n$.

* See Burnside and Panton, *Theory of Equations*. 1892. Page 430.

Reciprocally the condition of $g_1, g_2, \dots, g_n$ being all positive is also sufficient that ϕ be continuously positive for real values of the variables, and that ϕ be equal to zero only when all the variables vanish.

15. In order to have in as definite a form as possible the expression of ϕ as a sum of squares, we shall give to the expression [26] for g still a third form.

In connection with [12] it follows from [27] that

$$c_\nu = \sum_{\mu=1}^{\mu=n} A_{\nu\mu} e_\mu \quad (\nu=1, 2, \dots, n).$$

Denote by A the determinant of these equations, which from [20] is not identically zero, *i. e.*,

$$[30] \quad A = \sum \pm A_{11} A_{22} \dots A_{nn}.$$

We have as the solution of the preceding equation

$$e_\mu = \frac{1}{A} \sum_{\lambda=1}^{\lambda=n} \frac{\partial A}{\partial A_{\lambda\mu}} c_\lambda \quad (\mu=1, 2, \dots, n).$$

It follows from this in connection with [26] that

$$[31] \quad g = \frac{A}{\sum_{\lambda, \mu} \frac{\partial A}{\partial A_{\lambda\mu}} c_\lambda c_\mu},$$

an expression in which the c 's are subject only to the one condition

that
$$\sum \frac{\partial A}{\partial A_{\lambda\mu}} c_\lambda c_\mu \text{ is not identically zero.}$$

16. It shall next be shown that we may separate from $\phi(x_1, x_2, \dots, x_n)$ the square of a single variable in such a way that the resulting function contains only $n-1$ variables.

For example, in order that the expression $\phi - g x_n^2$ be expressed as a function of $n-1$ variables, we have chosen for g the value [31], after we have written in this expression $c_\lambda = 0$ ($\lambda = 1, 2, \dots, n-1$), while to c_n is given the value *unity*.

From this we have

$$g = \frac{A}{\frac{\partial A}{\partial A_{nn}}} = \frac{A}{A_1},$$

where A_1 is the determinant of the quadratic form $\phi(x_1, x_2, \dots, x_{n-1}, 0)$. Of course, this determinant must be different from zero.

Hence we may write

$$\phi(x_1, x_2, \dots, x_n) = \frac{A}{A_1} x_n^2 + \overline{\phi}(x'_1, x'_2, \dots, x'_{n-1}),$$

where

$$\overline{\phi}(x'_1, x'_2, \dots, x'_{n-1}) = \phi\left(x_1 - \frac{e_1}{e_n} x_n, x_2 - \frac{e_2}{e_n} x_n, \dots, x_{n-1} - \frac{e_{n-1}}{e_n} x_n, 0\right).$$

We may then proceed with $\overline{\phi}$ just as has been done with ϕ by separating the square of x'_{n-1} , etc.

If we notice that the determinant of the the function of $n-\mu$ variables which results from the separation of μ squares is the same as the determinant of the function which results from the original function, when we cause the μ last variables to vanish (in this function), and if we denote this determinant by A_μ , we have the following expression for ϕ :

$$\phi(x_1, x_2, \dots, x_n) = \frac{A}{A_1} x_n^2 + \frac{A_1}{A_2} x_{n-1}^2 + \dots + \frac{A_{n-2}}{A_{n-1}} \left(x_2^{(n-2)}\right)^2 + A_{n-1} \left(x_1^{(n-1)}\right)^2.$$

17. If now ϕ is to be continuously positive and equal to zero only when all the variables vanish, the coefficients on the right-hand side of the above expression must all be greater than zero. We therefore have the theorem:

In order that the quadratic form

$$\phi(x_1, x_2, \dots, x_n) = \sum_{\lambda, \mu} A_{\lambda\mu} x_\mu x_\lambda, \quad A_{\lambda\mu} = A_{\mu\lambda},$$

be a definite form and remain continuously positive, it is necessary and sufficient that the quantities $A_1, A_2, \dots, A_{n-1}$, which are defined through the equation $A_\mu = \sum \pm A_{11} A_{22} \dots A_{n-\mu, n-\mu}$

be all positive and different from zero. If on the other hand the quadratic form is to remain continuously negative, then of the quantities $A_{n-1}, A_{n-2}, \dots, A_1, A$, the first must be negative, and the following must be alternately positive and negative.

APPLICATION OF THE THEORY OF QUADRATIC FORMS TO THE PROBLEM OF MAXIMA AND MINIMA STATED IN ARTS. 1—6.

18. By establishing the criterion of the previous article the original investigation regarding the maximum and minimum of the function $f(x_1, x_2, \dots, x_n)$ is finished. The result established in art. 12 may in accordance with the definitions given in art. 7 be expressed as follows: *in order that a maximum and minimum of the function $f(x_1, x_2, \dots, x_n)$ may in reality enter on the position $(a_1, a_2, \dots, a_n)$ which is determined through the equations [6], it is necessary, if the second derivatives of the function do not all vanish at this position, that the aggregate of the terms of the second degree of the equation [4] be a DEFINITE quadratic form.* We can more especially say, as already indicated in art. 8: *If we have a definite form which only vanishes when all the variables vanish, the function has on the position in question in reality a maximum or minimum value; if, however, the form vanishes for other values of the variables, then a determination as to whether a maximum or minimum in reality exists, is not effected in the manner indicated and requires further investigation as is seen below.*

In accordance with the theorem stated in art. 8, a maximum or minimum will enter for a system of real values of the equation [6], if the homogeneous function of the second degree

$$\sum_{\lambda, \mu} \left\{ \left(\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\lambda \partial x_\mu} \right)_{a_1, a_2, \dots, a_n} h_\lambda h_\mu \right\},$$

or,

$$\sum_{\lambda, \mu} \left\{ f_{\lambda, \mu}(a_1, a_2, \dots, a_n) h_\lambda h_\mu \right\}$$

is a definite quadratic form; that is (art. 17) *there will be a minimum on the position $(a_1, a_2, \dots, a_n)$ if the quotients*

$$\frac{F_0}{F_1}, \frac{F_1}{F_2}, \dots, \frac{F_{n-2}}{F_{n-1}},$$

where $F_\mu = \sum \pm f_{11} f_{22} \dots f_{n-\mu, n-\mu}$, are all positive, a maximum, if they are all negative; and if in both cases the quotients are different from zero.

This last condition is only another form of what was said above, viz., that $\sum_{\lambda, \mu} f_{\lambda\mu} h_\lambda h_\mu$ cannot be a definite quadratic form, which vanishes for other values of the variables that are not all zero. This condition is in general necessary. For, if (say)

$$F_0 = \sum \pm f_{11} f_{22} \dots f_{nn} = 0,$$

then, the summation $\sum_{\lambda, \mu} f_{\lambda\mu} h_\lambda h_\mu$ being denoted by $\phi(h_1, h_2, \dots, h_n)$, this equation would directly imply the existence of a relation of the form:

$$\sum_{\nu} k_\nu \phi_\nu(h_1, h_2, \dots, h_n) = 0,$$

where the k_ν are constants which do not all simultaneously vanish.

If, therefore, k_n (say) is different from zero, we may write

$$\phi_n = -\frac{1}{k_n} \sum_{\nu=1}^{\nu=n-1} k_\nu \phi_\nu,$$

and with the help of this relation we have from the equation

$$\phi(h_1, h_2, \dots, h_n) = \sum_{\lambda=1}^{\lambda=n} \phi_\lambda(h_1, h_2, \dots, h_n) h_\lambda = \phi_n h_n + \sum_{\lambda=1}^{\lambda=n-1} \phi_\lambda h_\lambda$$

the following relation

$$\phi(h_1, h_2, \dots, h_n) = \sum_{\lambda=1}^{\lambda=n-1} \phi_\lambda(h_1, h_2, \dots, h_n) \left(h_\lambda - \frac{k_\lambda}{k_n} h_n \right).$$

Now in this expression we may so choose the arbitrary quantities h , that

$$h_\lambda = \frac{k_\lambda}{k_n} h_n \quad (\lambda=1, 2, \dots, n),$$

and consequently the function $\phi(h_1, h_2, \dots, h_n)$ would vanish without all the h 's becoming simultaneously zero.

Hence with this system of values of the h 's the difference $f(x_1, x_2, \dots, x_n) - f(a_1, a_2, \dots, a_n)$ would begin with terms of the third dimensions; and consequently the function $f(x_1, x_2, \dots, x_n)$ would have no maximum or minimum on the position $(a_1, a_2, \dots, a_n)$ unless all the terms of the third dimension vanish, a further condition being also that the aggregate of the terms of the fourth dimension have a continuously negative or a continuously positive sign.

We are not able to give the criterion for this; it is, however, the more improbable that such a case happens, the greater the number of variables that appear, since the necessary conditions (that the terms of the third dimension shall all vanish) continuously increase.

The problem of this chapter is thus completely treated; however the conditions that a quadratic form shall be a definite one, appear in a less symmetric form than we wish. It is due to the fact that we have given special preponderance to certain variables over the others.

We shall consequently take up the same subject again in the next chapter.

CHAPTER III.

THEORY OF MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES THAT ARE SUBJECTED TO SUBSIDIARY CONDITIONS.

1. In the preceding investigations the variables $x_1, x_2, \dots, x_n$ were completely independent of each other.

We now propose the problem: *Among all systems of values $(x_1, x_2, \dots, x_n)$ find those which cause the function $F(x_1, x_2, \dots, x_n)$ to have maximum and minimum values and which at the same time satisfy the equations of conditions:*

$$[1] \quad f_\lambda(x_1, x_2, \dots, x_n) = 0 \quad (\lambda = 1, 2, \dots, m; m < n),$$

where $f_\lambda(x_1, x_2, \dots, x_n)$ are functions of the same character as $f(x_1, x_2, \dots, x_n)$ in Chapt. II., art. 2.

2. The most natural way to solve the problem is to express by means of equations [1] m of the variables in terms of the remaining $n-m$ variables and write their values in $F(x_1, x_2, \dots, x_n)$. This function would then depend only upon the $n-m$ variables which are completely independent of one another, and so the present problem would be reduced to the one of the preceding chapter.

This method of procedure in practice, however, cannot be performed in most cases, since it is not always possible by means of equations [1] to represent in reality m variables as functions of the $n-m$ remaining variables. We must therefore seek a more practicable method.

3. If $(a_1, a_2, \dots, a_n)$ is any system of values of the quantities $x_1, x_2, \dots, x_n$ which satisfy the equations [1], then of the systems of values in the neighborhood of the position

$$(x_1 = a_1 + h_1, x_2 = a_2 + h_2, \dots, x_n = a_n + h_n),$$

we can consider only those which satisfy the equations [1], for which therefore we have

The equation of condition $f(x_1, x_2, x_3)=0$ represents then a surface upon which the point (x_1, x_2, x_3) , having the orthogonal coordinates x_1, x_2 and x_3 , is to lie, and for which $F(x_1, x_2, x_3)$ is to have a maximum or minimum value.

The determinants of the first order in the development

$$f(a_1+h_1, a_2+h_2, a_3+h_3)-f(a_1, a_2, a_3)$$

with respect to powers of h_1, h_2 and h_3 cannot all be equal to zero; that is, all the terms of the first dimension cannot vanish, the single terms being these determinants; and this means that the surface $f=0$ cannot have a singularity at the point in question.

Take next two equations of condition $f_1=0$ and $f_2=0$ between three variables x_1, x_2 and x_3 . Considered together, they represent a curve and the condition that the corresponding determinants of the second order cannot all be zero, means here that the curve at the point in question cannot have a singularity.

6. We may therefore assume that a determinant of the m th order of the system [4] is different from zero. Thus the developments [5] are possible. These power-series satisfy identically the equations [2] and offer, if the quantities $k_1, k_2, \dots, k_{n-m}$ are taken sufficiently small, all the values of the m quantities h_λ which likewise do not exceed certain limits. Suppose that the values of the m quantities h_λ are substituted in the difference

$$F(x_1, x_2, \dots, x_n) - F(a_1, a_2, \dots, a_n),$$

an expression which then depends only upon the $n-m$ variables $k_1, k_2, \dots, k_{n-m}$ that are independent of one another, and may consequently for sufficiently small values of these variables be developed in the form

$$\begin{aligned} [6] \quad & F(x_1, x_2, \dots, x_n) - F(a_1, a_2, \dots, a_n) \\ &= \sum_{\rho=1}^{\rho=n-m} e_\rho k_\rho + \frac{1}{2} \sum_{\rho, \sigma} e_{\rho\sigma} k_\rho k_\sigma + \dots \end{aligned}$$

We saw (Chapt. II., art. 6) that in order to have a maximum or minimum on the position $(a_1, a_2, \dots, a_n)$, it is necessary that the terms of the first dimension vanish, and consequently

$$[7] \quad e_\rho = 0 \quad (\rho=1, 2, \dots, n-m).$$

7. This condition may be easily expressed in another manner: For we obtain the quantities e , if in the development

$$F(x_1, x_2, \dots x_n) - F(a_1, a_2, \dots a_n) \\ = \sum_{\mu=1}^{\mu=n} F_{\mu} h_{\mu} + \frac{1}{2} \sum_{\mu, \nu} F_{\mu \nu} h_{\mu} h_{\nu} + \dots, \dots,$$

we substitute in the terms of the first dimension the values of the m quantities from [5] and arrange the result according to the quantities $k_1, k_2, \dots k_{n-m}$. In other words, the equations [7] ex-

press the condition that $\sum_{\mu=1}^{\mu=n} F_{\mu} h_{\mu}$ must vanish identically for all systems of values of the h 's, that satisfy the m equations [3] after they have been reduced to their linear terms. These are the m equations

$$[8] \quad \sum_{\mu=1}^{\mu=n} f_{\lambda \mu} h_{\mu} = 0 \quad (\lambda=1, 2, \dots m).$$

Now multiply the m equations respectively by the m arbitrary quantities $e_1, e_2, \dots e_m$, and adding the results to the equation

$$\sum_{\mu=1}^{\mu=n} F_{\mu} h_{\mu} = 0,$$

we have the following equation:

$$[9] \quad \sum_{\mu=1}^{\mu=n} \left\{ (F_{\mu} + e_1 f_{1\mu} + e_2 f_{2\mu} + \dots + e_m f_{m\mu}) h_{\mu} \right\} = 0.$$

But we may so determine the e 's that those terms in this summation drop out which contain the m quantities h , which are expressed in [5] through the $n-m$ other h 's; by causing these terms to vanish we have a system of m linear equations, whose determinant by hypothesis is different from zero.

Since the terms which remain of equation [9] are multiplied by the completely arbitrary quantities $k_1, k_2, \dots k_{n-m}$, it is not possible for this equation to exist unless each of the single coefficients is equal to zero.

Consequently we have as the first necessary condition for the appearance of a maximum or minimum the existence of the following system of n equation :

$$F_{\mu} + e_1 f_{1\mu} + e_2 f_{2\mu} + \dots + e_m f_{m\mu} = 0, \quad (\mu=1, 2, \dots, n)$$

and indeed in the sense that if m of these equation exist independently of one another, the remaining $n-m$ of them must be identically satisfied through the substitution of the e 's which are derived from the first m equation, it being of course presupposed that the system of values $(a_1, a_2, \dots, a_n)$ has already been so chosen that the equations [1] are satisfied.

Taking everything into consideration we may say : *In order that the function $F(x_1, x_2, \dots, x_n)$ have a maximum or minimum on any position $(a_1, a_2, \dots, a_n)$, it is necessary that the $n+m$ equations*

$$[10] \quad \left\{ \begin{array}{l} \frac{\partial F}{\partial x_{\mu}} + e_1 \frac{\partial f_1}{\partial x_{\mu}} + e_2 \frac{\partial f_2}{\partial x_{\mu}} + \dots + e_m \frac{\partial f_m}{\partial x_{\mu}} = 0, \\ \qquad \qquad \qquad (\mu=1, 2, \dots, n) \\ f_{\lambda}(x_1, x_2, \dots, x_n) = 0 \quad (\lambda=1, 2, \dots, m), \end{array} \right.$$

be satisfied by a system of real values of the $n+m$ quantities $x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_m$.

8. These deductions are connected with the one assumption that at least one of the determinants of the m th order which can be formed out of the $m \cdot n$ quantities [4] through the omission of $n-m$ columns does not vanish. This condition was necessary both for the determination of the quantities h , which satisfy the equations [2] and also for the determination of the m factors $e_1, e_2, \dots, e_m$.

It may happen that a maximum or minimum of the function F enters on the position $(a_1, a_2, \dots, a_n)$ even when the above condition is not satisfied. For if it is possible in any way to determine all systems of values of the h 's not exceeding certain limits that satisfy the equations [2], the equations [7] together with the equations [1] are sufficient in number to determine the n quantities $a_1, a_2, \dots, a_n$.

When the above condition is not satisfied, the equations [8] exist identically and consequently the equations [3], which serve

to determine the h 's begin with terms of the second dimension. We may often in this case proceed advantageously by introducing in the place of the original variables a system of $n-m$ new variables so chosen that when they are substituted in the given equations of condition, they identically satisfy them.

9. To make clear what has been said, the following example will be of service; its general solution is given in the sequel (Chapt. III., art. 28).

The problem proposed is: *find the shortest line which can be drawn from a given point to a given surface.* Upon the surface there are certain points of such a nature that the lines joining these points with the given point have the desired property and besides stand normal to the surface at these points.

If by chance it happens that one of these points is a double point (node) of the surface, so that at it we have $f_1=0$, $f_2=0$, $f_3=0$, then in reality for this point the terms of the first dimension in the equations [2] drop out and we have the case just mentioned.

If the surface is the right cone:

$$f(x, y, z) = 0 = x^2 + y^2 - z^2,$$

then we may write:

$$[11] \quad \begin{cases} x=2uv, \\ y=u^2-v^2, \\ z=u^2+v^2. \end{cases}$$

The equation of the surface is identically satisfied, and it is easily seen that we may express the quantities h_1, h_2, h_3 through two quantities k_1 and k_2 independent of each other even in the case where the required point of the surface is the vertex of the cone, that is, the point $x=0=y=z$, or $u=0=v$. This representation may indeed be effected in such a way that not only infinitely small values of h_1, h_2, h_3 correspond to infinitely small values of k_1, k_2 , but also that all systems of values h_1, h_2, h_3 are had which satisfy the equation

$$f(x+h_1, y+h_2, z+h_3) = 0.$$

We have, however, to give to the variables at one time real, at another time purely imaginary values, if the equation [11] is to represent the entire surface of the cone; but in this manner the

unavoidable trouble has taken such a direction that the proposed problem falls into two similar parts, which may be treated in full after the methods of Chapt. II. In other cases we may proceed in a similar manner. The special problem will each time of itself offer the most propitious method of procedure.

10. We must now establish the criteria from which one can determine whether a maximum or minimum of $F(x_1, x_2, \dots, x_n)$ really enters or not on a definite position $(a_1, a_2, \dots, a_n)$, which has been determined in accordance with the theorem cited in art. 7 of the present chapter.

One might consider this superfluous, since in virtue of the criteria given in the previous chapter a maximum or minimum will certainly enter, if the aggregate of terms of the second dimension in [6] is a definite quadratic form of the nature indicated.

It is, however, desirable to determine the existence of a maximum or minimum without having previously made the development of the function in the form [6]; since, in order to obtain the coefficients $e_{\rho\sigma}$, we must pay attention not only to the terms of the first dimension but also to the terms of the second dimension, when the values of [5] are substituted in the development of

$$F(x_1, x_2, \dots, x_n) - F(a_1, a_2, \dots, a_n) = \sum_{\mu=1}^{\mu=n} F_{\mu} h_{\mu} + \frac{1}{2} \sum_{\mu, \nu} F_{\mu\nu} h_{\mu} h_{\nu} + \dots$$

11. The above difficulty may be avoided, if we multiply each of the expressions [2] which vanish identically by the quantities e_{μ} ($\mu=1, 2, \dots, m$) respectively, add them thus multiplied to the above difference and then develop the whole expression with respect to the powers of h .

Owing to equation [9] terms of the first dimension can no longer appear in this development, and we have, if we write

$$[12] \quad F + \sum_{\mu=1}^{\mu=m} e_{\mu} f_{\mu} = G,$$

$$[13] \quad F(x_1, x_2, \dots, x_n) - F(a_1, a_2, \dots, a_n) = G(x_1, x_2, \dots, x_n) \\ - G(a_1, a_2, \dots, a_n) = \frac{1}{2} \sum_{\mu, \nu} G_{\mu\nu} h_\mu h_\nu + \dots$$

We thus have the homogeneous function of the second degree

$$\sum_{\rho, \sigma} e_{\rho\sigma} k_\rho k_\sigma \text{ of the formula [6], if we substitute in } \sum_{\mu, \nu} G_{\mu\nu} h_\mu h_\nu$$

the values [5] and consider only the terms of the first dimension in the process. If we then apply the criteria of the preceding chapter, we can determine whether the function F possesses or not a maximum or minimum on the position $(a_1, a_2, \dots, a_n)$.

12. The definite conditions that have been thus derived are unsymmetric for a two-fold reason: on the one hand because in the determination of the quantities h some of them have been given preference over the others, and on the other hand because those expressions, by means of which it is to be decided whether the function of the second degree is continuously positive or continuously negative, have been formed in an unsymmetric manner from the coefficients of the function.

It is therefore interesting to derive a criterion which is free from these faults, and which also indicates in many cases how the results will turn out. With this in view let us return to the problem already treated in the preceding chapter and propose the following more general theorem in quadratic forms.

THEORY OF HOMOGENEOUS QUADRATIC FORMS.

13. *Theorem.* We have given a homogeneous function of the second degree

$$[14] \quad \phi(x_1, x_2, \dots, x_n) = \sum_{\lambda, \mu} A_{\lambda\mu} x_\lambda x_\mu \\ (A_{\lambda\mu} = A_{\mu\lambda})$$

in n variables, which are subjected to the linear homogeneous equations of condition

$$[15] \quad \theta_\lambda = \sum_{\mu=1}^{\mu=n} a_{\lambda\mu} x_\mu = 0 \quad (\lambda=1, 2, \dots, m; \quad m < n);$$

we are required to find the conditions under which ϕ is continuously positive or continuously negative for all those systems of values of the variables which satisfy equations [15].

It is in every respect sufficient to solve this theorem with the limitation that the quantities x are subjected to the further condition :

$$[16] \quad x_1^2 + x_2^2 + \dots + x_n^2 = 1;$$

since, if ϕ is a definite quadratic form and positive, say, it is also a definite positive form for those values of $x_1, x_2, \dots, x_n$ which satisfy the equation [16].

Reciprocally, if $\phi(x_1, x_2, \dots, x_n)$ is positive for all the arguments which satisfy the equation [16], then it is also positive for all arbitrary arguments.

For let $x_1, x_2, \dots, x_n$ be an arbitrary system of values and form the expressions

$$\xi_i = \frac{x_i}{\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}} \quad (i=1, 2, \dots, n).$$

Noting that $\xi_i (i=1, 2, \dots, n)$ satisfy the equation [16] and writing $s = 1/\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, it is seen that

$$\phi(x_1, x_2, \dots, x_n) = s^2 \phi(\xi_1, \xi_2, \dots, \xi_n).$$

Hence $\phi(x_1, x_2, \dots, x_n)$ is positive when $\phi(\xi_1, \xi_2, \dots, \xi_n)$ is positive.

It is therefore in every respect admissible to add the equation [16]. We have, however, thereby gained an essential advantage: for owing to the condition [16] none of the variables can lie without the interval $-1 \dots +1$; further since the function varies in a continuous manner, it must necessarily have an upper and a lower limit for these values of the variables $x_1, x_2, \dots, x_n$; that is, among all systems of values which satisfy the equations [15] and [16] their must necessarily be one for which there is a maximum and one which gives a minimum value of ϕ .

We limit ourselves to the determination of the latter.

14. Through the addition of equation [16] we have reduced the theorem of the preceding article to a problem in the theory of maxima and minima; for, if the minimum value of $\phi(x_1, x_2, \dots, x_n)$ is positive, ϕ is certainly a definite positive form.

If we write

$$[17] \quad G = \phi - e \left(\sum_{\lambda=1}^{\lambda=n} x_{\lambda}^2 - 1 \right) + 2 \sum_{\rho=1}^{\rho=m} e_{\rho} \theta_{\rho},$$

then, in order to find the position at which there is a minimum value of the function, we have to form the system of equations

$$\frac{\partial G}{\partial x_{\lambda}} = 0 \quad (\lambda=1, 2, \dots, n).$$

This gives

$$\frac{\partial \phi}{\partial x_{\lambda}} - 2e x_{\lambda} + 2 \sum_{\rho=1}^{\rho=m} e_{\rho} \frac{\partial \theta_{\rho}}{\partial x_{\lambda}} = 0, \\ (\lambda=1, 2, \dots, n)$$

or,

$$[18] \quad \sum_{\mu=1}^{\mu=n} A_{\lambda \mu} x_{\mu} - e x_{\lambda} + \sum_{\rho=1}^{\rho=m} e_{\rho} a_{\rho \lambda} = 0. \\ (\lambda=1, 2, \dots, n)$$

From the $n+m+1$ equations

$$[19] \quad \left\{ \begin{array}{l} \sum_{\mu=1}^{\mu=n} A_{\lambda \mu} x_{\mu} - e x_{\lambda} + \sum_{\rho=1}^{\rho=m} e_{\rho} a_{\rho \lambda} = 0, \quad (\lambda=1, 2, \dots, n) \\ \theta_{\rho} = \sum_{\mu=1}^{\mu=n} a_{\rho \lambda} x_{\mu} = 0, \quad (\rho=1, 2, \dots, m) \\ \sum_{\lambda=1}^{\lambda=n} x_{\lambda}^2 = 1; \end{array} \right.$$

the $n+m+1$ quantities $x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_m, e$ may be determined. Since we know *a priori* that a minimum value of the function ϕ in reality exists on one position, we are certain that this system of equations must determined at least one real system of values.

Consequently the first $n+m$ linear homogeneous equations of [19]^e are consistent with one another, and may be solved with respect to the unknown quantities $x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_m$; their determinant must therefore vanish, and we must have

$$[20] \Delta e \equiv \begin{vmatrix} A_{11}-e, & A_{12}, \dots, A_{1n}, & a_{11}, \dots, a_{m1} \\ A_{21} & , & A_{22}-e, \dots, A_{2n}, & a_{12}, \dots, a_{m2} \\ \dots & \dots & \dots & \dots \\ A_{n1} & , & A_{n2}, \dots, A_{nn}-e, & a_{1n}, \dots, a_{mn} \\ a_{11} & , & a_{12}, \dots, a_{1n}, & 0, \dots, 0 \\ \dots & \dots & \dots & \dots \\ a_{m1} & , & a_{m2}, \dots, a_{mn}, & 0, \dots, 0 \end{vmatrix} = 0.$$

The equation $\Delta e = 0$ is clearly of the $n-m$ th degree in e . The minimum value of ϕ is necessarily contained among the roots of this equation; for if we multiply the equations [18] respectively by $x_1, x_2, \dots, x_n$ and add the results, we have

$$[21] \quad \phi(x_1, x_2, \dots, x_n) = e,$$

it being presupposed that the system of values $(x_1, x_2, \dots, x_n)$ together with the quantities $e_1, e_2, \dots, e_m$ satisfies the system of equation [19], which is only possible if e is a root of the equation $\Delta e = 0$. Further among the systems of values x which satisfy the system of equations [19], that system is also to be found which calls for the minimum, and since the value of the function which belongs to such a system of values is always a root of equation [20], it follows also that the required minimal value of ϕ must be contained among the roots of this equation.

As already remarked this minimal value must be positive if ϕ is to be continuously positive for the systems of values of the x 's under consideration, and from this it follows that Δe must have only positive roots. For if one root of this equation was negative, then for this root we could determine a system of values $x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_m$ for which as seen from [21] ϕ is likewise negative.

*Hence in order that ϕ be continuously positive for all systems of values of the x 's which satisfy the equations [15], it is necessary and sufficient that the equation $\Delta e = 0$ have only positive roots.**

The question next arises when does the equation $\Delta e = 0$ have only positive roots. It may be answered in a completely

* See Zajackowski, Annals of the Scientific Society of Cracow. Vol. XII, 1867

rigorous manner by means of Sturm's theorem*; but the investigation is somewhat difficult, and the symmetry, which we especially wish to preserve, would be lost, when we applied Sturm's theorem.

For develop the determinant according to powers of e as follows :

$$[22] \quad e^{n-m} - B_1 e^{n-m-1} + B_2 e^{n-m-2} - \dots + (-1)^{n-m} B_{n-m} = 0;$$

then, if all the roots of this equation are real and positive, the coefficients B must be all positive and ≤ 0 , and, reciprocally, if the roots of this equation are real and the B 's are all > 0 , then also the roots of the equation $\Delta e = 0$ are all positive.

16. We shall first show that all the roots of the equation $\Delta e = 0$ are real for the case where no equations of conditions are present.

Equation [20] reduces then to the form :

$$[23] \quad \begin{vmatrix} A_{11}-e & A_{12} & , & \dots & A_{1n} \\ A_{21} & , & A_{22}-e & , & \dots & A_{2n} \\ . & . & . & . & . & . \\ A_{n1} & , & A_{n2} & , & \dots & A_{nn}-e \end{vmatrix} = 0,$$

an equation, which is called the "EQUATION OF SECULAR VARIATIONS" and plays an important role in many analytical investigations, for example, in the determination of the secular variations of the orbits of the planets, as well as in the determination of the principal axes of lines and surfaces of the second degree.†

*Hermite, Crelle, bd. 52, p. 43;
Serret, Algèbre Sup. 1866, t. 1, p. 581;
Kronecker, Berlin Monatsbericht, 1873. Feb.

† In this connection the reader is referred to :
Laplace, Mém. de Paris. 1772. II., pp. 293-363;
Euler, Mém. de Berlin. 1749-50; Theoria motus corp. sol. 1765. Chapt. V;
Lagrange, Mém de Berlin. 1773, p. 108;
Poison et Hachette, Journ. de l'Ecole Polytechn. Cah. 11, p. 170. 1802;
Kummer, Crelle, bd. 26, p. 268;
Jacobi, Crelle, bd. 30, p. 46;
Bauer, Crelle, bd. 71, p. 40;
Borchardt, Liouv. Journ. t. 12, p. 30;
Sylvester, Phil. Mag. 1852. II., p. 138; etc.

17. Weierstrass' proof*, which is very simple, that all the roots of this equation are real, depends only upon the theorem that, if the determinant of a system of n homogeneous equations vanishes, it is always possible to satisfy the equations through values of the unknown quantities that are not all equal to zero.

Instead of the equation [16] we subject the variables to the somewhat more general equation

$$\psi(x_1, x_2, \dots, x_n) = 1,$$

where ψ denotes a homogeneous function of the second degree, which is always positive† and is only $= 0$ when the variables themselves vanish.

18. If we form the system of equations

$$[24] \quad \phi_\lambda - e \psi_\lambda = 0 \quad (\lambda = 1, 2, \dots, n),$$

then these equations may always be solved, if their determinant vanishes.

This determinant is exactly the same as that in [23] for the case that

$$\psi = \sum_{\lambda=1}^{\lambda=n} x_\lambda^2.$$

We assume that $e = k + l i$, and that we have found

$$x_\lambda = \xi_\lambda + \eta_\lambda i \quad (\lambda = 1, 2, \dots, n)$$

as a system of values that satisfy the equations [24].

We must consequently have

$$\phi_\lambda(\xi_1 + \eta_1 i, \xi_2 + \eta_2 i, \dots, \xi_n + \eta_n i) - (k + l i) \psi_\lambda(\xi_1 + \eta_1 i, \xi_2 + \eta_2 i, \dots, \xi_n + \eta_n i) = 0. \quad (\lambda = 1, 2, \dots, n)$$

Since the real and the imaginary parts of these equations must of themselves be zero, it follows, when we observe that ϕ_λ and ψ_λ are linear functions of the variables, that :

$$\phi_\lambda(\xi_1, \xi_2, \dots, \xi_n) - k \psi_\lambda(\xi_1, \xi_2, \dots, \xi_n) + l \psi_\lambda(\eta_1, \eta_2, \dots, \eta_n) = 0,$$

$$\phi_\lambda(\eta_1, \eta_2, \dots, \eta_n) - k \psi_\lambda(\eta_1, \eta_2, \dots, \eta_n) + l \psi_\lambda(\xi_1, \xi_2, \dots, \xi_n) = 0.$$

* Weierstrass, Berlin. Monatsbericht. 1868. May 18 ;

Cf. also Kronecker, Berlin. Monatsbericht. 1874, p. 1

† That we are justified in making this assumption, is seen from the *lemma*, which for the sake of continuity in the argument is not inserted until the end of the present discussion*(arts. 21, 22 and 23).

19. Multiply these equations respectively by η_λ and ξ_λ , take the summation over them from 1 to n , and subtracting one of the resulting equations from the other, then, since

$$\sum_{\lambda} \eta_{\lambda} \phi_{\lambda}(\xi_1, \xi_2, \dots, \xi_n) = \sum_{\lambda} \xi_{\lambda} \psi_{\lambda}(\eta_1, \eta_2, \dots, \eta_n),$$

$$\sum_{\lambda} \eta_{\lambda} \psi_{\lambda}(\xi_1, \xi_2, \dots, \xi_n) = \sum_{\lambda} \xi_{\lambda} \phi_{\lambda}(\eta_1, \eta_2, \dots, \eta_n),$$

we have

$$l \left\{ \sum_{\lambda} \eta_{\lambda} \psi_{\lambda}(\eta_1, \eta_2, \dots, \eta_n) + \sum_{\lambda} \xi_{\lambda} \psi_{\lambda}(\xi_1, \xi_2, \dots, \xi_n) \right\} = 0,$$

or,

$$[25] \quad l \left\{ \psi(\eta_1, \eta_2, \dots, \eta_n) + \psi(\xi_1, \xi_2, \dots, \xi_n) \right\} = 0.$$

If it is possible to find systems of values of the quantities $x_1, x_2, \dots, x_n$ which satisfy the equation [24] under the assumption that $e = k + li$, then these values must satisfy at the same time [25]; but since after our hypothesis the quantity within the brackets cannot vanish, it follows that l must be equal to zero; that is, every value of e for which the determinant vanishes, is real.

Hence we have the theorem :

In order that a quadratic form $\phi(x_1, x_2, \dots, x_n)$ be continuously positive, it is necessary and sufficient that the development of the determinant [23] which admits of only real roots, when expanded in powers of e , viz.—

$$[26] \quad e^n - B_1 e^{n-1} + B_2 e^{n-2} - \dots + (-1)^n B_n = 0,$$

consist of $n+1$ terms and that these terms be alternately positive and negative.

If the function is to be continuously negative, then the equation [26] must be complete and have continuation of sign.

20. Lemma. If a homogeneous function of the second degree $\psi(x_1, x_2, \dots, x_n)$ can become zero for any system of real values of the variables which are not all zero, then ψ may be both positive and negative, it being presupposed that the determinant of ψ is different from zero.

Let the function ψ vanish for the system of values $(\xi_1, \xi_2, \dots, \xi_n)$ and instead of $x_1, x_2, \dots, x_n$ write in ψ the arguments $\xi_1 + c_1 k, \xi_2 + c_2 k, \dots, \xi_n + c_n k$, where the c 's are indeterminate constants.

Developing with respect to powers of k we have

$$\psi(\xi_1 + c_1 k, \xi_2 + c_2 k, \dots, \xi_n + c_n k) = 2k \sum_{a=1}^{a=n} c_a \psi_a(\xi_1, \xi_2, \dots, \xi_n) + k^2 \psi(c_1, c_2, \dots, c_n) \dots \dots (i).$$

By hypothesis the ξ 's are not all zero, and the determinant of ψ being different from zero, it follows that ψ_a ($a=1, 2, \dots, n$) can not all be zero, since otherwise the equations $\psi_a = 0$ could be satisfied by the ξ 's in question, and consequently the determinant would vanish, which is contrary to our supposition. Since further c_a ($a=1, 2, \dots, n$) are arbitrary constants, we may so

choose them that $\sum_{a=1}^{a=n} c_a \psi_a(\xi_1, \xi_2, \dots, \xi_n)$ is not equal to zero.

Now by taking k sufficiently small we may cause the sign of the expression (i) to depend only upon the first term on the right-hand side of that expression.

Hence if we choose k positive or negative, we have systems of values $(x_1, x_2, \dots, x_n)$ which make ψ positive or negative.

21. The determinant of the system of equations [24] is formed from the partial derivatives of

$$\phi(x_1, x_2, \dots, x_n) - e\psi(x_1, x_2, \dots, x_n),$$

that is, from

$$\phi_a(x_1, x_2, \dots, x_n) - e\psi_a(x_1, x_2, \dots, x_n) = 0, \dots \dots \dots (ii) \\ (a=1, 2, \dots, n)$$

where ϕ_a and ψ_a denote $\frac{1}{2} \frac{\partial \phi}{\partial x_a}$ and $\frac{1}{2} \frac{\partial \psi}{\partial x_a}$, respectively. If this

determinant is equal to zero for a value of e , this means: we can give to the variables $x_1, x_2, \dots, x_n$ values that are not all zero and in such a way that the n equation (ii) exist. Let this value of e be $e = k + l i$; then if $l \leq 0$, we shall show that the function ψ can have both positive and negative values.

Denote the system of values $(x_1, x_2, \dots, x_n)$ which satisfy the equation (ii) by

$$x_a = \xi_a + i \eta_a \quad (a=1, 2, \dots, n);$$

then, as in art. 19, it may be proved that

$$l[\psi(\xi_1, \xi_2, \dots, \xi_n) + \psi(\eta_1, \eta_2, \dots, \eta_n)] = 0 \dots \dots (iii)$$

Since by hypothesis l is not zero, the equation (iii) can only exist either, when $\psi(\xi_1, \xi_2, \dots, \xi_n)$ and $\psi(\eta_1, \eta_2, \dots, \eta_n)$ have opposite values, and then it is proved, what we wish to show, that ψ can have both positive and negative values; or, when the two values of the function are both zero, and then from what was seen in the preceding article, ψ can take both positive and negative values.

22. In the same connection it is interesting to prove the following theorem: *If the determinant formed from the partial derivatives of the homogeneous quadratic form $\psi(x_1, x_2, \dots, x_n)$ is different from zero, and if among the infinite number of quadratic forms:*

$$\lambda \phi(x_1, x_2, \dots, x_n) + \mu \psi(x_1, x_2, \dots, x_n),$$

there is one definite quadratic form, the determinant formed from the partial derivatives of

$$\phi(x_1, x_2, \dots, x_n) - e \psi(x_1, x_2, \dots, x_n)$$

vanishes for only real values of e .

The theorem will also be true, if the determinant of ϕ (and not as assumed of ψ) is different from zero.

Let $\lambda_1 \phi + \mu_1 \psi$ be a definite quadratic form, and write

$$\lambda_1 \phi + \mu_1 \psi = \overline{\psi}(x_1, x_2, \dots, x_n).$$

We shall further choose two constants λ_0 and μ_0 in such a way that, when we put

$$\lambda_0 \phi + \mu_0 \psi = \overline{\phi}(x_1, x_2, \dots, x_n),$$

$\overline{\phi}$ is different from zero.

We know from the previous article that the determinant formed from the equations

$$\overline{\phi}_a - k \overline{\psi}_a = 0 \quad (a=1, 2, \dots, n)$$

can only vanish for real values of k .

The equations

$$\bar{\phi}_\alpha - k \bar{\psi}_\alpha = 0 \quad (\alpha=1, 2, \dots, n) \dots\dots\dots (iv)$$

may be written in the form

$$(\lambda_0 - k \lambda_1) \phi_\alpha + (\mu_0 - k \mu_1) \psi_\alpha = 0 \quad (\alpha=1, 2, \dots, n),$$

or

$$\phi_\alpha = \frac{k \mu_1 - \mu_0}{\lambda_0 - k \lambda_1} \psi_\alpha \dots\dots\dots (v)$$

$$(\alpha=1, 2, \dots, n)$$

If we eliminate $(x_1, x_2, \dots, x_n)$ from these equations, we must have the same determinant for their solution as from the equations (iv).

Hence every k which causes this last determinant to vanish, must also cause the first determinant to vanish. But the k 's are all real. Hence, if we form from them the n expressions

$$e = \frac{k \mu_1 - \mu_0}{\lambda_0 - k \lambda_1},$$

these quantities must also be real.

Hence the determinant of the n equations

$$\phi_\alpha - e \psi_\alpha = 0 \quad (\alpha=1, 2, \dots, n)$$

has always n real roots e .

We may therefore say: *If among all the quadratic forms which are contained in the form*

$$\lambda \phi(x_1, x_2, \dots, x_n) + \mu \psi(x_1, x_2, \dots, x_n),$$

there is one which can have only positive, or only negative values, then the determinant of $\phi - e \psi$ will have only real roots, it being assumed that the determinant of ϕ , or of ψ , is not zero.

The theorem in art. 17 is accordingly proved in its greatest generality.

23. The case where equations of condition are present may be easily reduced to the case already considered. The determinant [20] was the result of eliminating the quantities $x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_m$ from the $n + m$ equations:

$$[18] \quad \sum_{\mu=1}^{\mu=n} A_{\lambda \mu} x_{\mu} - e x_{\lambda} + \sum_{\rho=1}^{\rho=m} e_{\rho} a_{\rho \lambda} = 0 \quad (\lambda=1, 2, \dots, n),$$

$$[15] \quad \theta_{\rho} = \sum_{\mu=1}^{\mu=n} a_{\rho \mu} x_{\mu} = 0 \quad (\rho=1, 2, \dots, m).$$

Since the result of the elimination is independent of the way in which it has been effected, we may first consider m of the quantities x , say: $x_1, x_2, \dots, x_m$ expressed by means of the equations [15] in terms of the remaining $n-m$ of the x 's, which may be denoted by $\xi_1, \xi_2, \dots, \xi_{n-m}$. We thus have

$$[27] \quad x_{\mu} = \sum_{\nu=1}^{\nu=n-m} C_{\mu \nu} \xi_{\nu} \quad (\mu=1, 2, \dots, m).$$

Through the substitution of these values, let $\phi(x_1, x_2, \dots, x_n)$ be transformed into $\bar{\phi}(\xi_1, \xi_2, \dots, \xi_{n-m})$ and the equation

$$\sum_{\lambda=1}^{\lambda=n} x_{\lambda}^2 = 1, \text{ into } \psi(\xi_1, \xi_2, \dots, \xi_{n-m}) = 1.$$

The function ψ is continuously positive, and is only equal to zero, when the variables themselves all vanish.

The equations [18] may be written in the form:

$$\frac{1}{2} \frac{\partial \phi}{\partial x_{\lambda}} - e x_{\lambda} + \sum_{\rho=1}^{\rho=m} e_{\rho} \frac{\partial \theta_{\rho}}{\partial x_{\lambda}} = 0 \quad (\lambda=1, 2, \dots, n).$$

Multiply these equations respectively by $\frac{\partial x_{\lambda}}{\partial \xi_{\nu}}$ ($\lambda=1, 2, \dots, n$), and adding the results, then since

$$\sum_{\lambda=1}^{\lambda=n} \frac{\partial \theta_{\rho}}{\partial x_{\lambda}} \frac{\partial x_{\lambda}}{\partial \xi_{\nu}} = \frac{\partial \theta_{\rho}}{\partial \xi_{\nu}},$$

$$\sum_{\lambda=1}^{\lambda=n} \frac{\partial \phi}{\partial x_{\lambda}} \frac{\partial x_{\lambda}}{\partial \xi_{\nu}} = \frac{\partial \phi}{\partial \xi_{\nu}},$$

$$\sum_{\lambda=1}^{\lambda=n} x_{\lambda} \frac{\partial x_{\lambda}}{\partial \xi_{\nu}} = \frac{1}{2} \frac{\partial}{\partial \xi_{\nu}} \sum_{\lambda=1}^{\lambda=n} x_{\lambda}^2,$$

we have the following equations:

$$\frac{1}{2} \frac{\partial \phi}{\partial \xi_{\nu}} - \frac{1}{2} e \frac{\partial}{\partial \xi_{\nu}} \sum_{\lambda=1}^{\lambda=n} x_{\lambda}^2 + \sum_{\rho=1}^{\rho=m} e_{\rho} \frac{\partial \theta_{\rho}}{\partial \xi_{\nu}} = 0. \\ (\nu=1, 2, \dots, n)$$

The last term of this equation drops out, if we substitute in it the expressions [27], since the θ_{ρ} expressed in the ξ 's vanish identically, and we have the equations

$$[28] \quad \frac{\partial \phi}{\partial \xi_{\nu}} - e \frac{\partial \psi}{\partial \xi_{\nu}} = 0. \quad (\nu=1, 2, \dots, n-m).$$

Now give to the ν all values from 1 to $n-m$, and we have a system of $n-m$ linear homogeneous equations, from which we may eliminate the yet remaining $\xi_1, \xi_2, \dots, \xi_{n-m}$. The result of this elimination, which is an equation in e , must agree with [20]. The equations [28] are, however, created in exactly the same manner as the equations [24]. If then Δe is the determinant of these equations, it follows that the roots of the equation $\Delta e = 0$ are all real.

24. As the solution of the theorem proposed in art. 13, the final result is:

In order that the homogeneous function of the second degree

$$\phi(x_1, x_2, \dots, x_n) = \sum_{\lambda, \mu} A_{\lambda \mu} x_{\lambda} x_{\mu}, \\ A_{\lambda \mu} = A_{\mu \lambda},$$

be continuously positive for all systems of values of the quantities $x_1, x_2, \dots, x_n$, which satisfy the m linear homogeneous equations of condition,

$$\theta_{\rho} = \sum_{\mu} a_{\rho \mu} x_{\mu} = 0 \quad (\rho=1, 2, \dots, m),$$

it is necessary and sufficient that the form of the equation [20], developed with respect to powers of e and which has only real roots, consist of $n-m+1$ terms and that the signs associated with these terms be alternately positive and negative. There must, however, be only a continuation of sign, if ϕ is to be continuously negative.

The above method was first discovered by Lagrange, who did not, however, sufficiently emphasize the reality of the roots of equation [20].

APPLICATION OF THE CRITERIA JUST FOUND TO THE PROBLEM OF THIS CHAPTER.

25. We have determined the exact conditions that a homogeneous quadratic form be definite for the case where the variables are to satisfy equations of conditions, and in a manner entirely symmetric in the coefficients of the given function together with those of the given equations of condition.

At the same time with the solution of this problem, the problem of maxima and minima which we have proposed in this chapter, is solved.

26. Having regard to the remarks made in art. 7 and art. 10 we have as a final result of our investigations the following theorem.

Theorem. If those positions are to be found on which a given regular function $F(x_1, x_2, \dots, x_n)$ has a maximum or minimum value under the condition that the n variables $x_1, x_2, \dots, x_n$ satisfy the m equations

$$[a] \quad f_\lambda(x_1, x_2, \dots, x_n) = 0 \quad (\lambda = 1, 2, \dots, m),$$

where f_λ are likewise regular functions, we write

$$[b] \quad F + \sum_{\rho=1}^{\rho=m} e_\rho f_\rho = G(x_1, x_2, \dots, x_n),$$

and seek the system of real values

$$x_1, x_2, \dots, x_n, e_1, e_2, \dots, e_m,$$

which satisfy the $n+m$ equations

$$[c] \quad \begin{cases} \frac{\partial G}{\partial x_\lambda} = 0 & (\lambda=1, 2, \dots, n), \\ f_\lambda = 0 & (\lambda=1, 2, \dots, m). \end{cases}$$

If $(a_1, a_2, \dots, a_n)$ is such a system of values of $x_1, x_2, \dots, x_n$, then we develop the difference

$$G(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) - G(a_1, a_2, \dots, a_n)$$

with respect to powers of h , and have (since no terms of the first dimension can appear, owing to equations [c]) the following development:

$$[d] \quad G(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) - G(a_1, a_2, \dots, a_n) \\ = \frac{1}{2} \sum_{\mu, \nu} G_{\mu\nu}(a_1, a_2, \dots, a_n) h_\mu h_\nu + \dots$$

We must next see whether the function

$$[e] \quad \phi(h_1, h_2, \dots, h_n) = \sum_{\mu, \nu} G_{\mu\nu} h_\mu h_\nu$$

is continuously positive or continuously negative for all systems of values of the h 's which satisfy the m equations

$$[f] \quad \sum_{\mu=1}^{\mu=n} f_{\rho\mu}(a_1, a_2, \dots, a_n) h_\mu = 0 \quad (\rho=1, 2, \dots, m).$$

To do this we form the determinant

$$[g] \quad \begin{vmatrix} G_{11}-e, & G_{12}, \dots, G_{1n}, & f_{11}, f_{21}, \dots, f_{m1} \\ G_{21} & , & G_{22}-e, \dots, G_{2n}, & f_{12}, f_{22}, \dots, f_{m2} \\ . & . & . & . & . & . & . & . & . \\ G_{n1} & , & G_{n2}, \dots, G_{nn}-e, & f_{1n}, f_{2n}, \dots, f_{mn} \\ f_{11} & , & f_{12}, \dots, f_{1n}, & 0, 0, \dots, 0 \\ . & . & . & . & . & . & . & . & . \\ f_{m1} & , & f_{m2}, \dots, f_{mn}, & 0, 0, \dots, 0 \end{vmatrix},$$

and this determinant put $=0$ is an equation of the $m-n$ degree in e , which has only real roots. Developing the determinant with respect to powers of e , we have to see whether the development consists of $n-m+1$ terms with alternately positive and negative sign or with only continuation of sign.

If the first is the case, the function ϕ is continuously positive and the function F has on the position $(a_1, a_2, \dots, a_n)$ a MINIMUM value, if on the contrary the latter is true, then ϕ is continuously negative and F has on the position $(a_1, a_2, \dots, a_n)$ a MAXIMUM value.

This criterion fails, however, when ϕ vanishes identically, because the quantities G_μ vanish for the position $(a_1, a_2, \dots, a_n)$; and it also fails when the smallest or greatest root of $\Delta e = 0$ is zero, since in this case we may always so choose the h 's that ϕ vanishes without the h 's being all identically zero.

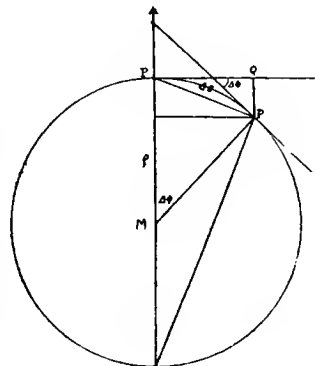
In both cases the development $[d]$ begins with terms of the third or higher dimensions, and for the same reason as that stated at the end of the last chapter we may assert that in general no maximum or minimum will enter on the position $(a_1, a_2, \dots, a_n)$.

We give next two geometrical examples illustrating the above principles.

27. PROBLEM I. Determine the greatest and the smallest curvature at a regular point of a surface.

If at a regular point P of a plane curve we draw a tangent and from a neighboring point P' on the curve we drop a perpendicular $P'Q$ upon this tangent, then the value that $2 \frac{P'Q}{PP'^2} = \frac{\Delta \phi}{\Delta s}$ approaches,

if we let P' come indefinitely near P , is called the *curvature* of the curve at the point P . If the curve is a circle with radius r , then the above ratio approaches $\frac{1}{r}$



as a limiting value, and is therefore the same for all points of the circle. Now construct the *osculating* circle which passes through the two neighboring points P and P' of the given curve. The arc of the circle PP' may be put equal to the arc PP' of the curve, when P and P' are taken very near each other, and consequently, if r is the radius of this circle, the curvature of the curve is determined through the formula

$$[1] \quad \frac{2 P'Q}{PP'^2} = \frac{1}{r}.$$

The quantity r is called the *radius of curvature* and the center M of the circle which lies on the normal drawn to the curve at the point P is known as the *center of curvature* at the point P . The curvature is counted positive or negative according as the line $P'Q$, or what amounts to the same thing, MP has the same or opposite direction as that direction of the normal which has been chosen positive.

If we have a given surface and if the normal at any regular point of this surface is drawn, then every plane drawn through this normal will cut the surface in a curve, which has at the point P a definite tangent and a definite curvature in the sense given above.

The *curvature* of this curve at the point P is called the *curvature* of the surface at the point $P=(x, y, z)$ in the direction of the tangent which is determined through the normal section in question.

Following the definitions given above it is easy to fix the analytic conception of the curvature of a surface and then to formulate the problem in an analytic manner:

If $P'=(x', y', z')$ is a neighboring point of P on the surface, the equation of the surface may be written in the form:

$$\begin{aligned}
 [2] \quad 0 = & F_1(x'-x) + F_2(y'-y) + F_3(z'-z) \\
 & + \frac{1}{2} \left\{ F_{11}(x'-x)^2 + F_{22}(y'-y)^2 + F_{33}(z'-z)^2 + 2F_{12}(x'-x)(y'-y) \right. \\
 & \left. + 2F_{23}(y'-y)(z'-z) + 2F_{31}(z'-z)(x'-x) \right\} + \dots,
 \end{aligned}$$

where

$$\begin{aligned}
 F_1 &= \frac{\partial F}{\partial x}, \quad F_2 = \frac{\partial F}{\partial y}, \quad F_3 = \frac{\partial F}{\partial z}, \\
 F_{11} &= \frac{\partial^2 F}{\partial x^2}, \quad F_{22} = \frac{\partial^2 F}{\partial y^2}, \quad F_{33} = \frac{\partial^2 F}{\partial z^2}, \\
 F_{12} &= \frac{\partial^2 F}{\partial x \partial y}, \quad F_{23} = \frac{\partial^2 F}{\partial y \partial z}, \quad F_{31} = \frac{\partial^2 F}{\partial z \partial x}.
 \end{aligned}$$

The equation of the tangential plane at the point P is

$$[3] \quad F_1(\xi-x) + F_2(\eta-y) + F_3(\zeta-z) = 0.$$

If we therefore write for brevity

$$[4] \quad \sqrt{F_1^2 + F_2^2 + F_3^2} = H,$$

and take as the positive direction of the normal of the surface at the point P that direction for which H is positive, then the direction-cosines of this normal are

$$\frac{F_1}{H}, \frac{F_2}{H} \text{ and } \frac{F_3}{H}.$$

Consequently the distance from P' to the tangential plane is

$$[5] \quad P'Q = \frac{F_1}{H}(x'-x) + \frac{F_2}{H}(y'-y) + \frac{F_3}{H}(z'-z).$$

The positive or negative sign is to be given to the expression on the right-hand side according as the length $P'Q$ has the same or opposite direction as that direction of the normal which has been chosen positive.

In the first case, paying attention to [2], which has to be satisfied since P' lies upon the surface, we have

$$[6] \quad \frac{2 P'Q}{PP'^2} = \frac{F_{11}(x'-x)^2 + F_{22}(y'-y)^2 + F_{33}(z'-z)^2 + 2F_{12}(x'-x)(y'-y) + \dots}{HS^2},$$

$$\text{where} \quad S^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2.$$

In the case where $P'Q$ is contrary to the positive direction of the normal, we must give the negative sign to the right-hand side of [6].

Now let P' approach nearer and nearer P , then the quantities

$$\frac{x'-x}{S}, \frac{y'-y}{S}, \frac{z'-z}{S},$$

which represent the direction-cosines of the line PP' become the direction-cosines of the tangent at the point P of the normal section that is determined through P' . Representing these by α , β , γ and the limiting value of $2 \frac{P'Q}{PP'^2}$ by κ , then is

$$[7] \quad \kappa = \frac{1}{H} \left\{ F_{11}\alpha^2 + F_{22}\beta^2 + F_{33}\gamma^2 + 2F_{12}\alpha\beta + 2F_{23}\beta\gamma + 2F_{31}\gamma\alpha \right\}.$$

In this formula κ represents the *curvature of the surface in the direction determined by α, β, γ* . This is to be taken positive or negative according as the direction of the length MP , where M is the center of curvature, corresponds to the positive direction or not.

If the coordinates of the center of curvature are represented by x_0, y_0, z_0 and the radius of curvature by ρ , then is

$$x - x_0 = \rho \frac{F_1}{H};$$

or, since $\kappa = \frac{1}{\rho}$,

$$[8] \quad \begin{cases} x - x_0 = \frac{F_1}{F_{11} \alpha^2 + F_{22} \beta^2 + \dots + 2F_{31} \gamma \alpha}, \\ y - y_0 = \frac{F_2}{F_{11} \alpha^2 + F_{22} \beta^2 + \dots + 2F_{31} \gamma \alpha}, \\ z - z_0 = \frac{F_3}{F_{11} \alpha^2 + F_{22} \beta^2 + \dots + 2F_{31} \gamma \alpha}. \end{cases}$$

Since H does not appear in these expressions, we see that the position of the center of curvature is independent of the choice of the direction of the normal.

Suppose that the normal plane which is determined through the direction α, β, γ is turned about the normal until it returns to its original position. Then while α, β, γ vary in a definite manner, the function κ of α, β, γ assumes different values at every instance, and since it is a regular function, it must have a maximum value for a definite system of values (α, β, γ) and likewise also a minimum value for another definite system of values (α, β, γ) .

The quantity $\frac{1}{H}$ has the same value for all normal sections that are laid through the same normal.* We have, therefore, to seek the systems of values (α, β, γ) for which the expression

$$F_{11} \alpha^2 + F_{22} \beta^2 + F_{33} \gamma^2 + 2F_{12} \alpha \beta + 2F_{23} \beta \gamma + 2F_{31} \gamma \alpha$$

assumes its greatest and its smallest value.

* See Salmon, A Treatise on the Analytic Geometry of Three Dimensions. Fourth Edition, p. 259.

We have also to observe that the variables α, β, γ must satisfy the equations of conditions:

$$[9] \quad \begin{cases} F_1 \alpha + F_2 \beta + F_3 \gamma = 0, \\ \alpha^2 + \beta^2 + \gamma^2 = 1; \end{cases}$$

the first of which says that the direction which is determined through α, β, γ is to lie in the tangential plane of the surface at the point P , while the second equation is the well known relation among the directions-cosines of a straight line in space.

Following the methods indicated in art. 26 we write

$$[10] \quad G = F_{11} \alpha^2 + F_{22} \beta^2 + \dots + 2F_{31} \gamma \alpha - e(\alpha^2 + \beta^2 + \gamma^2 - 1) + 2e'(F_1 \alpha + F_2 \beta + F_3 \gamma),$$

and we then have (art. 26, [c]) to form the equations:

$$\frac{\partial G}{\partial \alpha} = 0, \quad \frac{\partial G}{\partial \beta} = 0, \quad \frac{\partial G}{\partial \gamma} = 0, \\ F_1 \alpha + F_2 \beta + F_3 \gamma = 0;$$

from which we must eliminate α, β, γ and e' .

These equations are:

$$[11] \quad \begin{cases} (F_{11} - e) \alpha + F_{12} \beta + F_{13} \gamma + F_1 e' = 0, \\ F_{21} \alpha + (F_{22} - e) \beta + F_{23} \gamma + F_2 e' = 0, \\ F_{31} \alpha + F_{32} \beta + (F_{33} - e) \gamma + F_3 e' = 0, \\ F_1 \alpha + F_2 \beta + F_3 \gamma = 0, \end{cases}$$

where $F_{\lambda\mu} = F_{\mu\lambda}$ ($\lambda, \mu = 1, 2, 3$).

Through elimination we have

$$[12] \quad \begin{vmatrix} F_{11} - e & F_{12} & F_{13} & F_1 \\ F_{21} & F_{22} - e & F_{23} & F_2 \\ F_{31} & F_{32} & F_{33} - e & F_3 \\ F_1 & F_2 & F_3 & 0 \end{vmatrix} = 0.$$

This is an equation of the second degree in e , and consequently gives us two values e_1 and e_2 , which are maximum and minimum values, since both maximum and minimum values enter, as shown above. Multiply the three first equations [11] by α, β, γ respectively and adding the results, we have

$$[13] \quad F_{11} \alpha^2 + F_{22} \beta^2 + F_{33} \gamma^2 + 2F_{12} \alpha \beta + 2F_{23} \beta \gamma + 2F_{31} \gamma \alpha = e.$$

Hence, from [7] we have

$$[14] \quad \kappa = \frac{1}{\rho} = \frac{e}{H}.$$

Consequently the two principal curvatures at the point P have the values

$$[15] \quad \begin{cases} \frac{1}{\rho_1} = \frac{e_1}{H}, \\ \frac{1}{\rho_2} = \frac{e_2}{H}, \end{cases}$$

and the coordinates of the corresponding centres of curvature are found from the formulae

$$[16] \quad \begin{cases} x - x_{01} = \frac{F_1}{e_1}, & y - y_{01} = \frac{F_2}{e_1}, & z - z_{01} = \frac{F_3}{e_1}, \\ x - x_{02} = \frac{F_1}{e_2}, & y - y_{02} = \frac{F_2}{e_2}, & z - z_{02} = \frac{F_3}{e_2}. \end{cases}$$

In order to determine e , let us write

$$\begin{aligned} D_{11} &= (F_{22} - e)(F_{33} - e) - F_{23}^2 \\ D_{12} &= F_{23}F_{13} - F_{12}(F_{33} - e), \end{aligned}$$

and form from these the corresponding quantities through the cyclic interchange of the indices. Equation [12] may be written in the form*

$$D_{11}F_1^2 + D_{22}F_2^2 + D_{33}F_3^2 + 2D_{12}F_1F_2 + 2D_{23}F_2F_3 + 2D_{31}F_3F_1 = 0.$$

Developing this expression with respect to powers of e we have

$$[17] \quad H^2 e^2 - L e + M = 0,$$

$$\begin{aligned} \text{where} \quad L &= H^2 (F_{11} + F_{22} + F_{33}) - (F_{11}F_1^2 + F_{22}F_2^2 + F_{33}F_3^2) \\ &\quad + 2F_{12}F_1F_2 + 2F_{23}F_2F_3 + 2F_{31}F_3F_1 \end{aligned}$$

$$\begin{aligned} \text{and} \quad M &= (F_{22}F_{33} - F_{23}^2)F_1^2 + (F_{33}F_{11} - F_{31}^2)F_2^2 \\ &\quad + (F_{11}F_{22} - F_{12}^2)F_3^2 + (F_{12}F_{13} - F_{23}F_{11})F_2F_3 \\ &\quad + (F_{23}F_{21} - F_{31}F_{22})F_3F_1 + (F_{31}F_{32} - F_{12}F_{33})F_1F_2. \end{aligned}$$

* See Salmon, p. 257.

From [17] we have at once the values of the sum and the product of the two principal curvatures, viz.—(see equation 15):

$$[18] \quad \begin{cases} \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{L}{H^4}, \\ \frac{1}{\rho_1 \rho_2} = \frac{M}{H^4}. \end{cases}$$

We have thus expressed the sum of the reciprocal radii of curvature and also the measure of curvature of the surface at the point P directly through the coordinates of this point.

Although the formulae are somewhat complicated, they are used extensively and with great advantage.

In the case of *minimal surfaces**, which are characterized through the equation

$$\begin{aligned} \rho_1 + \rho_2 &= 0, \\ L &= 0. \end{aligned}$$

we have

This is therefore the general differential equation for minimal surfaces.

ART. 28. PROBLEM II. *From a given point (a, b, c) to a given surface $F(x, y, z) = 0$, draw a straight line whose length is a maximum or a minimum.*

Write

$$G = (x-a)^2 + (y-b)^2 + (z-c)^2 + 2\lambda F(x, y, z) \dots\dots\dots (i)$$

Then the quantities x, y, z, λ are to be determined (see [c] art. 26) from the following equations:

$$\begin{cases} x-a + \lambda F_1 = 0, \\ y-b + \lambda F_2 = 0, \\ z-c + \lambda F_3 = 0, \\ F(x, y, z) = 0. \end{cases} \dots\dots\dots (ii)$$

It follows, since F_1, F_2, F_3 are proportional to the direction-cosines of the normal to the surface at the point (x, y, z) , that the points determined through these equations are such that lines joining them to the point (a, b, c) stand normal to the surface.

If $P = (x, y, z)$ is such a point, then to determine whether for this point the quantity

$$(x-a)^2 + (y-b)^2 + (z-c)^2$$

* See papers on this subject in the first numbers of the *Mathematical Review*.

is in reality a maximum or a minimum, we substitute $x+u$, $y+v$, $z+w$ instead of x , y , z in the function G . The quantities u , v , w are of course taken very small.

We must develop the difference

$$G(x+u, y+v, z+w) - G(x, y, z) \dots \dots \dots (iii)$$

in powers of u , v and w .

The terms of the first dimension drop out and the aggregate of the terms of the second dimension is

$$\begin{aligned} \psi = & u^2 + v^2 + w^2 + \lambda (F_{11} u^2 + F_{22} v^2 + F_{33} w^2 \\ & + 2F_{12} uv + 2F_{23} vw + 2F_{31} wu) \dots \dots \dots (iv) \end{aligned}$$

Since the point $(x+u, y+v, z+w)$ must also lie upon the surface, the quantities u , v , w must satisfy the condition

$$F_1 u + F_2 v + F_3 w = 0, \dots \dots \dots (v)$$

where the terms of the higher dimensions are omitted (See [8] of the present chapter).

If we wish to determine whether the function ψ is continuously positive or continuously negative for all systems of values (u, v, w) which satisfy equation (v), we may seek the minimum or maximum of this function ψ under the condition that the variables are limited besides the equation (v) to the further restriction [cf. (16)] that

$$u^2 + v^2 + w^2 - 1 = 0 \dots \dots \dots (vi)$$

For this purpose we form the function

$$\psi - e(u^2 + v^2 + w^2 - 1) + 2e'(F_1 u + F_2 v + F_3 w), \dots \dots (vii)$$

and writing its partial derivatives with respect to u , v and w equal to zero, we derive the equations:

$$\left. \begin{aligned} \left(F_{11} - \frac{e-1}{\lambda}\right) u + F_{12} v + F_{13} w + \frac{e'}{\lambda} F_1 &= 0, \\ F_{21} u + \left(F_{22} - \frac{e-1}{\lambda}\right) v + F_{23} w + \frac{e'}{\lambda} F_2 &= 0, \\ F_{31} u + F_{32} v + \left(F_{33} - \frac{e-1}{\lambda}\right) w + \frac{e'}{\lambda} F_3 &= 0. \end{aligned} \right\} \dots \dots \dots (viii)$$

Eliminating u , v , w , $\frac{e'}{\lambda}$ from equations [v] and (viii) we have

here exactly the same system of equations as in [12] of the preceding problem, except that there $\frac{e-1}{\lambda}$ and e' stand in the place of e and e' .

Denote the two roots of the quadratic equation in e , which is the result of the above elimination, by e_1 and e_2 and the corresponding radii of curvature of the normal sections by ρ_1 and ρ_2 , then, since $\frac{e-1}{\lambda}$ has the same meaning as e in the previous problem:

$$\frac{1}{\rho_1} = \frac{e_1-1}{\lambda} \cdot \frac{1}{H},$$

$$\frac{1}{\rho_2} = \frac{e_2-1}{\lambda} \cdot \frac{1}{H},$$

where the positive direction of the normal to the surface is so chosen that $H > 0$.

If for the position (x, y, z) a minimum of the distance is to enter, then both values of the e must be positive, if a maximum, then e_1 and e_2 must be negative.

It is easy to give a geometric interpretation of the result just obtained: Let PN be the positive direction of the normal and $A = (a, b, c)$. Then from (ii) it follows that the length from A to P has the same or opposite direction as PN according as λ is negative or positive.

Hence from (ii)

$$AP = -\lambda H.$$

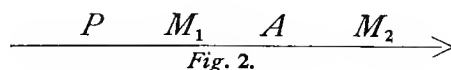
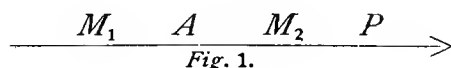
If the centres of curvature corresponding to ρ_1 and ρ_2 be denoted by M_1 and M_2 , then is

$$M_1 P = \frac{\lambda H}{e_1-1} = -\frac{AP}{e_1-1},$$

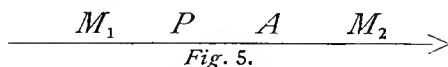
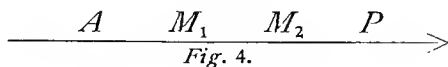
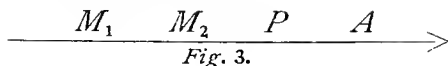
$$M_2 P = \frac{\lambda H}{e_2-1} = -\frac{AP}{e_2-1}.$$

$$\text{Hence } e_1 = \frac{M_1 A}{M_1 P} \text{ and } e_2 = \frac{M_2 A}{M_2 P}.$$

If then M_1 and M_2 lie on the same side of P and if A lies between M_1 and M_2 as in figs. 1 and 2, then the e 's have different signs and there is neither a maximum nor a minimum.



If M_1 and M_2 lie on the same side of P and A without the interval $M_1 \dots M_2$, then a maximum will enter according as A starting from one of the centres of curvature lies upon the same side as P or not (see figs. 3 and 4).



If the points M_1 and M_2 lie on different sides of P and if A is situated within the interval $M_1 \dots M_2$ as in Fig. 5, then there is always a minimum. If, however, A lies without the interval $M_1 \dots M_2$, then there is neither a maximum nor a minimum.

In whatever manner M_1 and M_2 may lie, if A coincides with one of these points, then one of the two values of e is equal to zero and then the general result stated at the end of art. 26 is applicable.

The case may also happen here [see art. 8 of this chapter] that in the solution of the equations (ii) and (iii) a singular point of the surface is found as the point P , at which $F_1 = 0 = F_2 = F_3$.

For such a case we cannot proceed as above; since, there being no definite normal of the surface at such a point, the determination whether for this point a maximum or minimum really exists, cannot be decided.

The general remark of art. 8 indicates how we are to proceed.

CHAPTER IV.

SPECIAL CASES.

THE PRACTICAL APPLICATION OF THE CRITERIA THAT HAVE BEEN HITHERTO GIVEN AND A METHOD FOUNDED UPON THE THEORY OF FUNCTIONS, WHICH OFTEN RENDERS UNNECESSARY THESE CRITERIA.

1. The practical application of the established criteria is in many cases connected with very great, if not insurmountable difficulties, which, however, cannot be disregarded in the theory. For often the solutions of the equations $[c]$, art. 26, of the previous chapter cannot be effected without great difficulty, if at all, and therefore also the formation of the function ϕ is impossible. It also happens, even if the function ϕ can be formed, that the discussion regarding the coefficients of $\Delta e = 0$ is attended with much difficulty. Moreover, the formation of the function ϕ and the investigation relative to the coefficients of Δe are very often unnecessary, since through direct observation we may in many cases determine whether a maximum or a minimum really exists. If it happens that the equations $[c]$ admit of only one real solution (*i.e.* of a real system of values $x_1, x_2, \dots, x_n$), then we may be sure that this is in reality the maximum or minimum of the function. In the same way, if we can convince ourselves *à priori* that a maximum and a minimum exist, and if it happens that the equations $[c]$ offer only two real systems of values, it is evident that the one system must correspond to the maximum of the function, the other system to the minimum value.

The determination which of the two systems of values gives the one or the other, is in most cases easily determined.

2. As has already been indicated in the introduction (see art. 1, Chapt. I.), one cannot be too careful in the investigation whether on a position which has been determined from the equations $[a]$ and $[c]$ of art. 26, Chapt. III. there really is a maximum

or minimum, since there are cases in which one may convince himself of the existence of a maximum or minimum, when in reality there is no maximum or minimum.

For example, to establish Euclid's theorem respecting parallel lines, one tries to prove the theorem regarding the sum of the angles of a triangle without the help of the theorem of the parallel lines. Legendre was able indeed to show that this sum could not be greater than two right angles; however, he did not show that they could not be less than two right angles. The method of reasoning employed at that time was as follows: If in a triangle the sum of the three angles cannot be greater than 180° , then there must be a triangle, for which the maximum of the sum of these angles is really reached. Assuming this to be correct, it may be shown that in this triangle the sum of the angles $= 180^\circ$, and from this, it may be proved that the same is true of all triangles.

We see at once that a fallacy has been made. For if we apply the same conclusions to the spherical triangles, in the case of which the sum of the angles cannot be smaller than 180° , we would find that in every spherical triangle the sum of the angles is $= 180^\circ$, which is not true.

The fallacy consists in the assumption of the existence of a maximum or minimum; it is not always necessary that an upper or lower limit be reached, even if one can come just as near to it as is wished.

On this account also in our case the assumption of the existence of a real maximum is not allowed without further proof. We therefore endeavor to give the *existence-proof*. For this purpose we must recall several theorems in the theory of functions*.

3. We call the collectivity of all systems of values, which n variable quantities $x_1, x_2, \dots, x_n$ can assume, the *realm* (Gebiet) of these quantities, and each single system of values a *position* in this realm. If these quantities are variables without restriction, so that each of them can go from $-\infty$ to $+\infty$, we call the realm considered as a whole (Gesamtgebiet) an *n-ple multiplicity* (n -fache Mannigfaltigkeit). For example, the points of a straight line in the plane form a *simple* (einfach) multiplicity. The points

* Cf. Chapt. I., especially arts. 16 et seq.

in a plane, a *double* multiplicity; in space, a *triple* multiplicity. All straight lines in space form a *4-ple* multiplicity. For consider two planes, then in each of them a point is determined by two quantities, and a straight line by two points, *i. e.*, by four quantities. Also all surfaces of spheres in space form a *4-ple* multiplicity, since every sphere needs for its complete determination the three coordinates of the centre and the radius of the sphere.

If the quantities are connected with one another by equations of condition, so that m of them (say) may be considered as independent, then we call the collectivity of all positions an *m-ple* multiplicity, or a structure* of the m th kind in the region of these n variables.

For example, a straight line in space is a structure of the first kind in the region of three variables. If $x_1, x_2, \dots, x_n$ are independent of one another, then we say a definite position ($a_1, a_2, \dots, a_n$) lies *on the interior of the realm*, if these positions and also all their neighboring positions belong to this region; it lies *upon the boundary of the realm*, if in each neighborhood as small as we wish of this position, there are present positions which belong to the realm, and also those that do not belong to it; it lies finally *without the defined realm*, if in no neighborhood as small as we wish of this position there are positions which belong to the defined region.

If the quantities $x_1, x_2, \dots, x_n$ are subjected to m equations of condition, then we may express these in terms of $n-m$ independent variables $u_1, u_2, \dots, u_{n-m}$, and the same definition may be applied to these variables.

4. The following theorems are proved in the theory of functions: (1)† If a continuous variable quantity is defined in any manner, this quantity has an *upper* and a *lower* limit; that is, there is a definitely determined quantity g of such a kind that no value of the variable can be greater than g , although there is a value of the variable, which can come as near to g as we wish. In the same way there is a quite determined quantity k of such a nature that no value of the variable is less than k , although there is a value of the variable that comes as near to k as we wish.

* See Chapt. I., art. 7 and art. 9. See also Chapt. I., art. 16.

† Dini, *Theorie der Functionen*, p. 68. See also a paper by Stolz, B. Bolzano's Bedeutung in der Geschichte der Infinitesimal Rechnung. *Math. Ann.*, bd. XVIII.

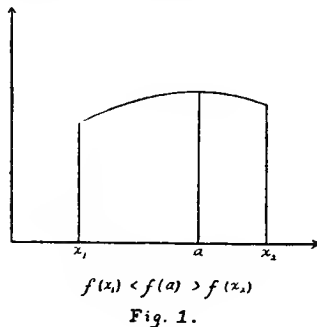
(2)* In the region of n variables $x_1, x_2, \dots, x_n$, suppose we have an infinite number of positions defined in any manner,—let these be denoted by $(x'_1, x'_2, \dots, x'_n)$,—further suppose that among the positions we have such positions, that x'_n can come as near to a fixed limit a_n as we wish. Then we have in the region of the quantities $x_1, x_2, \dots, x_n$ always at least one definite position $(a_1, a_2, \dots, a_n)$ of such a nature that among the definite positions $(x'_1, x'_2, \dots, x'_n)$ there are always present positions that lie as near this position as we wish, so that therefore, if δ denotes a quantity arbitrarily small,

$$|x'_\lambda - a_\lambda| < \delta \quad (\lambda=1, 2, \dots, n).$$

This position lies either within or upon the boundary of the defined region $(x'_1, x'_2, \dots, x'_n)$.

5. This presupposed, let us consider a continuous function $F(x_1, x_2, \dots, x_n)$, and let the realm of the quantities $x_1, x_2, \dots, x_n$ be a limited one, so that therefore, we have systems of values, which do not belong to it. If for every possible system of values $(x_1, x_2, \dots, x_n)$, we associate the corresponding value of the function, which may be denoted by x_{n+1} , then we have defined certain positions in the region of $n+1$ quantities. For the quantity x_{n+1} there is according to the first theorem an upper limit a_{n+1} ; consequently owing to the second theorem there must be within the interior or upon the limits of the defined region a position $(a_1, a_2, \dots, a_n, a_{n+1})$ of such a nature that in the neighborhood of this position there certainly exist positions, which belong to the region in question.

The case of a maximum.



Now if it can be shown that this position lies within the interior of the region, then there is in reality a maximum of the function on the position $(a_1, a_2, \dots, a_n)$; on the contrary, if the position lies on the boundary, then we cannot come to a conclusion regarding the existence of a maximum of the function x_{n+1} .

It may in many cases happen, that one can show, if $(x_1, x_2, \dots, x_n)$ is any

* Biermann, Theo. der An. Funk., p. 81; Serret, Diff. et Int. Cal., p. 26.

The case of a minimum.

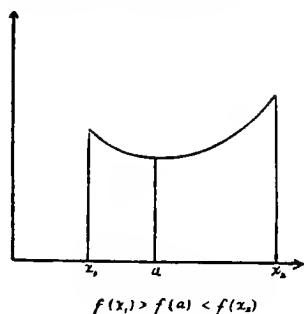


Fig. 2.

Case of asymptotic approach

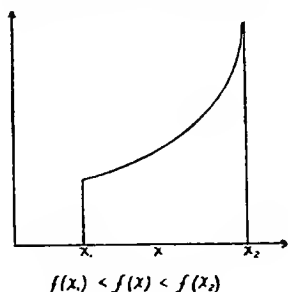


Fig. 3.

Maximum on the limiting position x_1 .

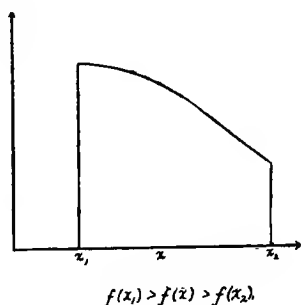


Fig. 4.

position on the boundary of the realm and if x_{n+1} denotes the corresponding value of the function, that there are present within the realm positions for which the values of the function are greater than for every position on the boundary. Then, of course, the position, which we are considering here, cannot lie upon the boundary, and it is clear that the limiting value of the function can be assumed for a definite position within the interior, since the function varies in a continuous manner. The analog is, of course, true for a minimum. If, however, it does not admit of proof that there are positions on the interior of the defined realm for which the value of the function is greater or smaller than it is for all positions on the boundary, then nothing can be concluded regarding the real existence of a maximum or minimum; the position $(a_1, a_2, \dots, a_n)$ would then lie on the boundary of the region, and there might be an asymptotic approach to the limiting value a_{n+1} without this value being in reality reached (cf. Chapt. I., art. 11). This, however, need not necessarily be the case.

The figures give a plain picture of what has been said for the case $y=f(x)$, where x is limited to the interval $(x_1 \dots x_2)$.

IT IS POSSIBLE THAT THE DIFFERENCE

$$F(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) - F(a_1, a_2, \dots, a_n)$$

IS NEITHER POSITIVE NOR NEGATIVE BUT ZERO ON

THE POSITION $(a_1, a_2, \dots, a_n)$ WHICH IS TO BE INVESTIGATED.

6. We shall now consider a case which is not included in the previous investigations but may be in a certain measure re-

duced to them: The definition of the maximum or minimum of a function consists in the fact that the difference

$$F(a_1+h_1, a_2+h_2, \dots, a_n+h_n) - F(a_1, a_2, \dots, a_n), \dots (i)$$

must be continuously negative or continuously positive. There are cases where a maximum or a minimum does not appear on the position $(a_1, a_2, \dots, a_n)$ in the sense that the above difference must be positive or negative, but in the sense that the difference must be zero.

Suppose, for example, we have the problem: Determine a polygon of n sides with a given constant perimeter S whose area is a maximum,—a problem which we shall later discuss more fully (see art. 10).

If this maximum is attained for a definite polygon, then we may at pleasure change the system of coordinates by sliding the polygon in the plane without altering the area.

For example, let $n=3$, and (x_1, y_1) , (x_2, y_2) and (x_3, y_3) be the coordinates of the vertices of the triangle. Then the expression which is to be a maximum is

$$F = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3),$$

where the variables are subjected to the condition

$$S = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} \\ + \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}.$$

There will not only be one system of values which gives for F a maximum value, but an infinite number of such positions; since, if we take the triangle in a definite position, we may move it in its plane at pleasure. This is therefore a case where the difference (i) is not positive or negative but zero.

7. Such cases, however, may be reduced to maxima and minima proper, if we choose arbitrarily some of the variable quantities. In the special example of the preceding article we may assume a vertex of the triangle at pleasure; let it be the origin of coordinates, and we further assume that one of the sides coincides with the positive direction of the X -axis, so that we may write

$x_1=y_1=z_2=0$. If we agree that the triangle is to lie above or below the X -axis, the problem is completely determinate.

8. In so far as the necessary conditions for the existence of a maximum or minimum are concerned, we may proceed in precisely the same manner as we have hitherto done; since under the assumption that there are no equations of condition, we have

$$\begin{aligned} & F(a_1+h_1, a_2+h_2, \dots, a_n+h_n) - F(a_1, a_2, \dots, a_n) \\ &= \sum_{a=1}^{a=n} h_a F_a(a_1, a_2, \dots, a_n) + (h_1, h_2, \dots, h_n)^{(2)} \dots \dots (ii) \end{aligned}$$

If a minimum is to be present, then this difference can never be negative, but may be zero. For this to be possible the first de-

rivatives must all vanish. Since, if the sum $\sum_{a=1}^{a=n} h_a F_a(a_1, a_2, \dots, a_n)$

had (say) a positive value for $h_1=c_1, h_2=c_2, \dots, h_n=c_n$, then we could place $h_a=c_a h$, and then choose h so small that the sign of the right-hand side of (ii) depends only upon the sign of the first term. If we then make h positive or negative the difference would also be positive or negative.

If equations of condition are present, it may be showed as above that the derivatives of the first order must vanish, since, if all these derivatives did not vanish, we might express some of the h 's through the remaining, and then proceed as we have just done. The required systems of values $(x_1, x_2, \dots, x_n)$ will therefore be determined from the same equations as before.

9. If we have found a system of values of the x 's which satisfy the equations of condition of the problem, then in the neighborhood of this position there will be an infinite number of other positions which satisfy the equations. These last are characterized by the condition that the difference (i) vanishes identically for them.

This is just the condition that made impossible the former criteria, by means of which we could decide whether a maximum or minimum really entered on a position $(a_1, a_2, \dots, a_n)$ that was determined through the equations in $x_1, x_2, \dots, x_n$.

One must therefore seek in another manner to convince himself which case is the one in question.

This is further discussed in the following example.

10. PROBLEM. *Among all polygons which have a given number of sides and a given perimeter, find the one which contains the greatest surface-area.*

We see at once that the problem proposed here is of a somewhat different nature than the problems of arts. 27 and 28 of the previous chapter, since the existence of the maximum value of the function is no longer the question as was proposed in art. 4 of Chapt. II. and held as fixed throughout the general discussions. For, if the definition of the maximum is such that the function on the position $(a_1, a_2, \dots, a_n)$ must have a greater value on this position than on all neighboring positions, then in this sense our polygon could certainly not have a maximum area: Since, if we had such a polygon on any position, we might slide the polygon at pleasure without changing its shape and consequently its area. Therefore, only a maximum of the area can enter in the sense that the periphery remaining the same, an increase in the area of the surface cannot enter for an infinitely small sliding of the end-points. We consequently cannot apply our general theory without further discussion.

11. Let the coordinates of the n end-points taken in a definite order be

$$x_1, y_1, x_2, y_2, \dots, x_n, y_n.$$

The double area of a triangle, which has the origin as one of its vertices and the coordinates of the other two vertices x_1, y_1 and x_2, y_2 is, neglecting the sign, determined through the expression

$$x_1 y_2 - x_2 y_1.$$

In order to determine the sign of this expression we suppose that the fundamental system of coordinates is brought through turning about its origin into such a position that the positive X -axis falls together with the length 01. We call that side of the line 01 *positive* which corresponds to the positive direction of the Y -axis: The double area of the triangle 012 is to be counted positive or negative, according as it lies on the positive or negative side of the line 01.

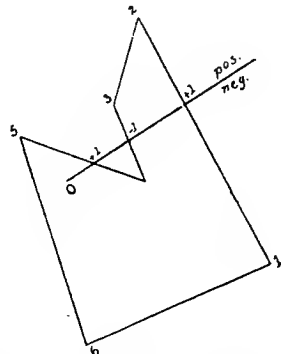
If the point 0 has the coordinates x_0, y_0 , the double area of the triangle is

$$2 \Delta_{012} = (x_1 - x_0)(y_2 - y_0) - (y_1 - y_0)(x_2 - x_0),$$

where the above criterion with reference to the sign is to be applied.

For the polygon we shall take a definite consecutive arrangement of the points $(1, 2, \dots, n)$ and besides shall assume that no two of the sides cross each other. The last hypothesis is justifiable, since we may easily convince ourselves that, if two sides cut each other, we may at once construct a polygon whose sides do not cut one another, and which having the same perimeter as the first polygon incloses a greater area.

Within the polygon take a point $0 = (x_0, y_0)$ and draw from it in any direction a straight line to infinity. This straight line always cuts an odd number of sides of the polygon.



Now if we follow the periphery of the polygon in the fixed direction $(1, 2, \dots, n)$ and mark the intersection of a side by the straight line with $+1$ or -1 , according as we pass from the negative to the positive side of that line or *vice versa*, then the sum of these marks is either $+1$ or -1 . In the first case we say that the polygon has been described in the positive direction, in the second case in the negative direction.

It may be proved* that, whatever point be taken as the point 0 within the polygon, and in whatever direction the straight line be drawn, we always have the same characteristic number $+1$ or -1 , if in each case the positive side of the straight line has been correctly determined.

12. The double area of the polygon is

$$\begin{aligned} 2F = & (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0) + (x_2 - x_0)(y_3 - y_0) \\ & - (x_3 - x_0)(y_2 - y_0) + \dots + (x_n - x_0)(y_1 - y_0) - (x_1 - x_0)(y_n - y_0); \\ \text{or, } \quad 2F = & x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + \dots + x_n y_1 - x_1 y_n \dots \dots (a) \end{aligned}$$

* The proof is found in Cremona, *Elementi di geometria proiettiva*. Rome, 1873.

where the positive or negative sign is to be taken according as the polygon has been described in the positive or negative direction. We may, however, eventually bring it about through reverting the order of the sequence of the end-points that the expression $2F$ is always positive.

13. Suppose that this has been done. The function $2F$ is to be made a maximum under the condition that the periphery has a definite value S .

We may write: $s_{1,2} + s_{2,3} + \dots + s_{n-1,n} + s_{n,1} = S, \dots \dots (\beta)$

where $s_{\lambda-1,\lambda} = \sqrt{(x_\lambda - x_{\lambda-1})^2 + (y_\lambda - y_{\lambda-1})^2} \dots \dots (\gamma)$

Form the function $G = 2F + e(s_{1,2} + s_{2,3} + \dots + s_{n,1} - S), \dots \dots (\delta)$

and placing its partial derivatives $= 0$, we have

$$\left. \begin{aligned} \frac{\partial G}{\partial x_\lambda} &= y_{\lambda+1} - y_{\lambda-1} + e \left(\frac{x_\lambda - x_{\lambda+1}}{s_{\lambda+1,\lambda}} + \frac{x_\lambda - x_{\lambda-1}}{s_{\lambda-1,\lambda}} \right) = 0, \\ \frac{\partial G}{\partial y_\lambda} &= -x_{\lambda+1} + x_{\lambda-1} + e \left(\frac{y_\lambda - y_{\lambda+1}}{s_{\lambda+1,\lambda}} + \frac{y_\lambda - y_{\lambda-1}}{s_{\lambda-1,\lambda}} \right) = 0 \end{aligned} \right\} \dots \dots (\epsilon)$$

($\lambda = 1, 2, \dots, n$; however for $\lambda = n$, we must write: $\lambda + 1 = 1$).

• • Take in addition the equation (β) and we have $2n + 1$ equations for the determination of the $2n + 1$ unknown quantities

$$x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, e.$$

14. In order to come in the simplest manner to the desired result from these equations, we adopt the following mode of procedure. If we write

$$z_\lambda = (x_\lambda - x_{\lambda-1}) + i(y_\lambda - y_{\lambda-1}), \dots \dots (\zeta)$$

then z_λ , geometrically interpreted, represents the length from the the point $\lambda - 1$ to the point λ both in value and in direction.

If further we write

$$z'_\lambda = (x_\lambda - x_{\lambda-1}) - i(y_\lambda - y_{\lambda-1}), \dots \dots (\eta)$$

then is $z_\lambda \cdot z'_\lambda = s_{\lambda-1,\lambda}^2 \dots \dots (\theta)$

Multiply the first of equations (ϵ) by i and subtract from the result the second, then owing to (ζ) we have

$$z_{\lambda} + z_{\lambda+1} + e i \left(\frac{z_{\lambda}}{s_{\lambda-1, \lambda}} - \frac{z_{\lambda+1}}{s_{\lambda, \lambda+1}} \right) = 0,$$

or

$$\left. \begin{aligned} z_{\lambda} \left(1 + \frac{e i}{s_{\lambda-1, \lambda}} \right) &= -z_{\lambda+1} \left(1 - \frac{i e}{s_{\lambda, \lambda+1}} \right) \\ z'_{\lambda} \left(1 - \frac{e i}{s_{\lambda-1, \lambda}} \right) &= -z'_{\lambda+1} \left(1 + \frac{e i}{s_{\lambda, \lambda+1}} \right) \end{aligned} \right\} \dots\dots\dots (\iota)$$

Now multiplying the last two equations together, we have from (θ)

$$s_{\lambda-1, \lambda}^2 \left(1 + \frac{e^2}{s_{\lambda-1, \lambda}^2} \right) = s_{\lambda, \lambda+1}^2 \left(1 + \frac{e^2}{s_{\lambda, \lambda+1}^2} \right),$$

and therefore $s_{\lambda-1, \lambda}^2 = s_{\lambda, \lambda+1}^2$.

15. Since $s_{\nu-1, \nu}$ is an essentially positive quantity, it follows that

$$s_{\lambda-1, \lambda} = s_{\lambda, \lambda+1} \dots\dots\dots (\kappa)$$

and consequently the sides of the polygon are all equal to one another. Hence each side $= \frac{S}{n}$, and we have from (ι)

$$\frac{z_{\lambda+1}}{z_{\lambda}} = \frac{e i n + s}{e i n - s} = \text{const.}$$

If we write $z_{\lambda} = \frac{S}{n} e^{\phi_{\lambda} i}$,

where ϕ_{λ} denotes the angle which $s_{\lambda-1, \lambda}$ makes with the X -axis,

then $e^{(\phi_{\lambda+1} - \phi_{\lambda}) i} = \text{const.},$

or $\phi_{\lambda+1} - \phi_{\lambda} = \text{const.}; \dots\dots\dots (\lambda)$

that is, all the angles of the polygon are equal to one another, and consequently the polygon is a regular one.

Weierstrass thus showed that the conditions which are had from the vanishing of the first derivatives can be satisfied only

by a regular polygon; that is, if there is a polygon, which, with a given perimeter and a prescribed number of sides, has a greatest area, this polygon is necessarily regular.

Our deductions, however, have in no manner revealed that a maximum really exists.

16. In order to prove the existence of a maximum we must apply the method given in arts. 2-5 of the present chapter. We see that a limit is given for the area of the polygon from the fact that the number of sides and the perimeter are given; for, if we consider a square whose sides are greater than the given perimeter S , we can lay each polygon with the perimeter S in this square, and in such a way that the end-points of the polygon do not fall upon the sides of the square. Hence, the area of the polygon cannot be greater than that of the square, and consequently there must be an upper limit for this area, which may be denoted by F_0 . It may yet be asked whether this limit can in reality be reached for a definite system of values. The variables $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ being limited to this square, there must be among the positions $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ which fill out the square a position $(a_1, b_1, a_2, b_2, \dots, a_n, b_n)$ of such a nature, that in every neighborhood of this position, other positions exist, for which the corresponding surface area F of the polygon formed from them comes as near as we wish to the upper limit. We may assume that this position is within the square, since if it lies by chance on the boundary, then from what has been said above, it is admissible to slide the corresponding polygon without altering its shape and area into the interior of the square.

We assert that the value of the function F for the position $(a_1, b_1, a_2, b_2, \dots, a_n, b_n)$ must necessarily be equal to F_0 . Since, if this was not the case, the inequality must also remain, if we subject the points $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ to an infinitely small variation; and on account of the continuity of F , it would not be possible in the arbitrary neighborhood of $(a_1, b_1, \dots, a_n, b_n)$ to give positions for which the corresponding area comes arbitrarily near the upper limit F_0 . This, however, contradicts the conclusions previously made. Hence, all n corners with a given periphery not only approach a definite limit with respect to their inclosed area but this limit is in reality reached. Since further the necessary conditions for the existence of a maximum have given the regular polygon of

n sides as the only solution, and since we have seen a maximum really exists, we may with all rigor make the conclusion: *That polygon, which, with a given periphery and a given number of sides, contains the greatest area, is the regular polygon.*

We have now given what we consider the general Weierstrassian theory of maxima and minima, and in the sequel we shall discuss a few special problems of maxima and minima.

CASES IN WHICH THE SUBSIDIARY CONDITIONS ARE NOT TO BE REGARDED AS EQUATIONS BUT AS LIMITATIONS.

17. Besides the problems already mentioned, those problems are particularly deserving of notice, in which the conditions for the variables are not given in the form of equations but as restrictions or limitations.

For example, let a point in space and a function which depends upon the coordinates of this point be given. Further let the point be so restricted that it always remains within the interior of an ellipsoid; then the restriction made upon the point is expressed through the inequality

$$0 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1.$$

We have accordingly such limitations when a function of the variables is given which cannot exceed a certain upper and a certain lower limit.

We make such a restriction when we assume that a function f_1 shall always lie between fixed limits a and b .

18. This limitation which consists of two inequalities

$$[a] \qquad a < f_1 < b$$

may be easily reduced to one.

For from $[a]$ it follows necessarily that

$$[\beta] \qquad \frac{f_1 - a}{b - f_1} > 0;$$

and reciprocally, if $[\beta]$ exists, and if $a < b$, then f_1 must be situated between a and b and consequently $[a]$ must be true.

Every limitation of the kind given may be analytically represented as one single inequality of the form $[\beta]$.

19. We must next find the algorithm for the cases under consideration. This may be done at once if we consider that such cases may be reduced to those in which occur equations of condition. For this purpose we need only establish the problem of finding the maximal or minimal values of a function whose variables are subjected to certain conditions as follows:

It is required among all systems of values which satisfy the equations $f_\lambda = 0$ ($\lambda = 1, 2, \dots, m$), to find those for which F is a maximum or a minimum.

By proposing the problem in this manner, it is clear that all the variables x which appear in the equations of condition need not necessarily be contained in the function.

Suppose further we have the limitation that

$$[\gamma] \qquad f_k > 0,$$

then, through the introduction of a new variable x_{n+1} we may transform this limitation into an equation of condition. For, as we have to do with only real values of the variables, the equation

$$[\gamma^a] \qquad f_k = x_{n+1}^2$$

denotes exactly the same thing as $[\gamma]$.

If therefore a function $F(x_1, x_2, \dots, x_n)$ is to be a maximum or minimum under the limitations

$$f_1 = 0, f_2 = 0, \dots, f_m = 0, f_{m+1} > 0, f_{m+2} > 0, \dots, f_{m+r} > 0,$$

where the f 's are functions of $x_1, x_2, \dots, x_n$, then we may solve this problem, if instead of the r last restrictions, we introduce the the following limitations:

$$f_{m+1} = x_{n+1}^2, f_{m+2} = x_{n+2}^2, \dots, f_{m+r} = x_{n+r}^2.$$

The problem is thus reduced to the one of finding among the systems of variables $x_1, x_2, \dots, x_{n+r}$, those systems for which F is a maximum or minimum.

20. Examples of this character occur very frequently in mechanics. As an example, consider *a pendulum which consists of a flexible thread that cannot be stretched*, then the condition under which the motion takes place is not that the material point remains at a constant distance from the origin, but that the dis-

tance cannot be greater than the length of the thread. Such problems are more closely considered in the sequel. It will be seen that by means of GAUSS' PRINCIPLE *all problems of mechanics may be reduced to problems of maxima and minima.*

GAUSS' PRINCIPLE.

21. For the sake of what follows, we shall give a short account of this principle: Consider the motion of a system of points whose masses are $m_1, m_2, \dots, m_n$. Let the motions of the points be limited or restricted in any manner, and suppose that the system moves under the influence of forces that act continuously. For a definite time, let the positions of the points and the components of velocity both in direction and magnitude be determined. The manner in which the motion takes place from this period on is determined through Gauss' Principle:

Let $A_1, A_2, \dots, A_n$, be the positions of the points at the moment first considered;

$B_1, B_2, \dots, B_n$, the positions which the points can take after the lapse of an infinitely small time τ , if the motions of these points are free;

$C_1, C_2, \dots, C_n$, the positions in which these points *really* are after the lapse of the same time τ ; and finally let

$C'_1, C'_2, \dots, C'_n$ be the positions which the points may also possibly have assumed after the time τ , when the conditions are fulfilled.

If we form

$$\sum_{\nu=1}^{\nu=n} m_{\nu} \overline{B_{\nu} C_{\nu}}^2 \quad \text{and} \quad \sum_{\nu=1}^{\nu=n} m_{\nu} \overline{B_{\nu} C'_{\nu}}^2,$$

it follows from Gauss' Principle, that from $\tau=0$ up to a definite value of τ , the condition

$$[1] \quad \sum_{\nu=1}^{\nu=n} m_{\nu} \overline{B_{\nu} C_{\nu}}^2 < \sum_{\nu=1}^{\nu=n} m_{\nu} \overline{B_{\nu} C'_{\nu}}^2$$

is always satisfied, that is, $\sum_{\nu=1}^{\nu=n} m_{\nu} \overline{B_{\nu} C_{\nu}}^2$ must always be a minimum.

22. In order to make rigorous conclusions from Gauss' Principle, which was briefly sketched in the preceding article, we shall give a more analytic formulation of it: For this purpose we denote the coordinates of A_v by x_v, y_v, z_v , the components of the velocity of A_v by x'_v, y'_v, z'_v , the components of the force acting upon A_v by X_v, Y_v, Z_v . The coordinates of B_v are therefore

$$x_v + \tau x'_v + \frac{\tau^2}{2} X_v, y_v + \tau y'_v + \frac{\tau^2}{2} Y_v, z_v + \tau z'_v + \frac{\tau^2}{2} Z_v;$$

and from Taylor's theorem the coordinates of C_v are

$$x_v + \tau x'_v + \frac{\tau^2}{2} x''_v + \dots, y_v + \tau y'_v + \frac{\tau^2}{2} y''_v + \dots, z_v + \tau z'_v + \frac{\tau^2}{2} z''_v + \dots;$$

consequently we have

$$[2] \sum m_v \overline{B_v C_v}^2 = \sum m_v \left\{ (x''_v - X_v)^2 + (y''_v - Y_v)^2 + (z''_v - Z_v)^2 \right\} \frac{\tau^4}{4} + \dots$$

Instead of x''_v , however, (see preceding art.) other values may possibly enter, say $x''_v + \xi_v, \dots$, so that we have

$$[3] \sum_{v=1}^{v=n} m_v \overline{B_v C'_v}^2 = \sum_{v=1}^{v=n} m_v \left\{ (x''_v + \xi_v - X_v)^2 + (y''_v + \eta_v - Y_v)^2 + (z''_v + \zeta_v - Z_v)^2 \right\} \frac{\tau^4}{4} + \dots$$

It follows from Gauss' Principle that the difference of the sums [2] and [3] must always be positive.

Hence

$$[4] \quad 0 > \sum_{v=1}^{v=n} m_v \left\{ 2 [\xi_v (x''_v - X_v) + \eta_v (y''_v - Y_v) + \zeta_v (z''_v - Z_v)] + \xi_v^2 + \eta_v^2 + \zeta_v^2 \right\} \frac{\tau^4}{4} + \dots;$$

that is, the quantities x''_v, y''_v, z''_v must be such that the sum [2] is a minimum.

Hence among all the x''_v , y''_v , z''_v which are associated with the conditions of motion, we must seek those which make [2] a minimum.

23. We have reached our proposed object, if we can show that the ordinary equations of mechanics may be derived from Gauss' Principle.

If there are no equations of condition present, then clearly [2] is only a minimum when

$$x''_v = X_v, \quad y''_v = Y_v, \quad z''_v = Z_v.$$

If, however, we have equations of condition, for example, $f(x, y, z) = 0$, then these must hold true throughout the whole motion. They may therefore be differentiated. We have in this way equations in $\frac{dx_v}{dt}$, $\frac{dy_v}{dt}$ and $\frac{dz_v}{dt}$. Differentiate again and we have equations in x''_v , y''_v and z''_v .

Hence, in conformity with the rules that have been hitherto found for the theory of maxima and minima, the quantities x''_v , y''_v , z''_v are to be so determined that the derived equations of condition are satisfied, while at the same time [2] becomes a minimum. But in this case also, as is easily shown, we are led to the usual differential equations of mechanics.

TWO APPLICATIONS OF THE THEORY OF MAXIMA AND MINIMA TO REALMS THAT SEEM DISTANT FROM IT.

I. *Cauchy's proof of the existence of the roots of algebraic equations.*

24. An interesting application of the theory of maxima and minima (cf. art. 5) is Cauchy's existence-proof of the roots of an algebraic equation:

Let $f(z)$ be an integral rational function of a real or complex variable z . The function becomes, when z is put $= x + iy$,

$$f(z) = \phi(x, y) + i\psi(x, y),$$

where ϕ and ψ are real functions of the real variables x and y .

Hence $\phi^2(x, y) + \psi^2(x, y)$ is always a positive quantity. The variables x and y may be arbitrarily great; but, if $x^2 + y^2$ —the square of the distance of the position (x, y) from the origin—is infinitely large, then $\phi^2 + \psi^2$ is infinitely large.

where $A_{\lambda i}$ is a homogeneous function of the i th degree in $x_1, x_2, \dots, x_n$ and consequently:

$$\begin{aligned} X_{\lambda} = & 2x_1 \frac{\partial A_{\lambda 2}}{\partial x_1} + 2x_2 \frac{\partial A_{\lambda 2}}{\partial x_2} + \dots + 2x_n \frac{\partial A_{\lambda 2}}{\partial x_n} \\ & + 3x_1 \frac{\partial A_{\lambda 3}}{\partial x_1} + 3x_2 \frac{\partial A_{\lambda 3}}{\partial x_2} + \dots + 3x_n \frac{\partial A_{\lambda 3}}{\partial x_n} \\ & + 4x_1 \frac{\partial A_{\lambda 4}}{\partial x_1} + 4x_2 \frac{\partial A_{\lambda 4}}{\partial x_2} + \dots + 4x_n \frac{\partial A_{\lambda 4}}{\partial x_n} \\ & + \dots \dots \dots; \\ & (\lambda=1, 2, \dots, n) \end{aligned}$$

or,
$$X_{\lambda} = x_1 X_{\lambda 1} + x_2 X_{\lambda 2} + \dots + x_n X_{\lambda n}.$$

The quantities $X_{\lambda \mu}$ ($\lambda=1, 2, \dots, n$; $\mu=1, 2, \dots, n$) are continuous functions of the x 's which become infinitely small with the x 's.

The system of equations [1] may be then brought to the form [1]

$$[1a] \quad y_{\lambda} = \sum_{\mu=1}^{\mu=n} \left(a_{\lambda \mu} + X_{\lambda \mu} \right) x_{\mu}. \quad (\lambda=1, 2, \dots, n)$$

The theorem of art. 25 in this modified form, may be expressed as follows:

(1) *It is always possible so to fix for the variables $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$, limits $g_1, g_2, \dots, g_n$ and $h_1, h_2, \dots, h_n$ that for every system of the y 's for which $|y_{\lambda}| < h_{\lambda}$ there exists ONE system of the x 's, for which $|x_{\lambda}| < g_{\lambda}$, and in such a way that the equations [1a] are satisfied.*

(2) *The solution of the equations [1a] has a similar form as the equations [1a] themselves, viz.:*

$$[3a] \quad x_{\lambda} = \sum_{\mu=1}^{\mu=n} \left(\frac{A_{\lambda \mu}}{A} + Y_{\lambda \mu} \right) y_{\mu}, \quad (\lambda=1, 2, \dots, n)$$

where the $Y_{\lambda\mu}$ are continuous function of the y 's which become infinitely small with these quantities.

To prove this theorem we make use of the theory of maxima and minima.

28. If we give to the y_λ the value zero, the equations [1a] can only be satisfied if their determinant vanishes, that is, when

$$[4] \quad \left| a_{\lambda\mu} + X_{\lambda\mu} \right| = 0, \\ (\lambda, \mu = 1, 2, \dots, n)$$

except for the case where the x 's vanish.

For sufficiently small values of the x 's, the determinant [4] is not very different from the determinant [2]. We may therefore determine limits g for the x 's, so that [4] cannot be zero unless [2] is also zero. $A=0$ is, however, by hypothesis excluded. Accordingly the y 's can only be zero in [1a] when all the x 's vanish. The x 's are thus confined within fixed limits which may be regarded as the boundaries of a definite realm.

29. Again we write

$$[5] \quad y_\lambda = \sum_{\mu=1}^{\mu=n} \left(a_{\lambda\mu} + X_{\lambda\mu} \right) x_\mu = F_\lambda (x_1, x_2, \dots, x_n), \\ (\lambda = 1, 2, \dots, n)$$

and consider the function

$$[6] \quad S = \sum_{\lambda=1}^{\lambda=n} F_\lambda^2 (x_1, x_2, \dots, x_n).$$

In S we will write for the x 's all the systems of values, where at least one x lies on the boundary of the realm in question. We regard as the realm of the x 's the totality of the x 's for which

$$\left| a_{\lambda\mu} + X_{\lambda\mu} \right| \text{ is only zero, when } \left| a_{\lambda\mu} \right| = 0; \\ (\lambda, \mu = 1, 2, \dots, n)$$

it follows then that [4] is not zero, since $|a_{\lambda\mu}|$ is by hypothesis different from zero.

In order to make this clear, it may be mentioned that as the boundaries of the realm, we must consider the totality of the x 's where at least one of the x 's reaches its limit.

For the limits, that is, when one of the x 's reaches its limit, there is no system of values of the x 's for which the function [6] vanishes, since the function can (as follows from definition [5] of the F_λ and the considerations of art. 28) only vanish, if all the y 's and consequently all the x 's vanish.

There is then a lower limit G which is different from zero for the values of [6], which correspond to a system of values $(x_1, x_2, \dots, x_n)$ of the limits.

30. We come next to the determination of the limiting values of the y 's. For this purpose we consider

$$[7] \quad \sum_{\lambda=1}^{\lambda=n} \left(F_\lambda (x_1, x_2, \dots, x_n) - y_\lambda \right)^2.$$

If we ascribe definite values to the y 's, then there is for the values [7] in the realm of the x 's, a system for which [7] is a minimum.

We wish to show that this system of values of the x 's does not lie upon the boundary of the realm. We prove this by showing that there is a point within the realm, where the expression [7] has a smaller value than it has on the boundary:

The expression [7] may be written

$$\sum_{\lambda=1}^{\lambda=n} (F_\lambda - y_\lambda)^2 = S - 2 \sum_{\lambda=1}^{\lambda=n} F_\lambda y_\lambda + \sum_{\lambda=1}^{\lambda=n} y_\lambda^2.$$

Since $\sqrt{S}$ is at all events greater than F_λ and consequently

$$\frac{F_\lambda}{\sqrt{S}} < 1,$$

it follows that

$$\begin{aligned} \sum_{\mu=1}^{\mu=n} F_\mu y_\mu &< \sqrt{S} \sum_{\mu=1}^{\mu=n} y_\mu < \sqrt{S} \left| \sum_{\mu=1}^{\mu=n} y_\mu \right| \\ &< \sqrt{S} \sum_{\mu=1}^{\mu=n} |y_\mu| < \sqrt{S} \sum_{\mu=1}^{\mu=n} h_\mu, \end{aligned}$$

where the h 's are the limits of the y 's. From this it results that

$$\sum_{\mu=1}^{\mu=n} (F_{\mu} - y_{\mu})^2 > S - 2 \sqrt{S} \sum_{\mu=1}^{\mu=n} h_{\mu} + \sum_{\mu=1}^{\mu=n} y_{\mu}^2,$$

and, consequently, for a greater reason

$$[8] \quad \sum_{\mu=1}^{\mu=n} (F_{\mu} - y_{\mu})^2 > S - 2 \sqrt{S} \sum_{\mu=1}^{\mu=n} h_{\mu}.$$

The limits have to be so chosen that the right-hand side of [8] is positive. This choice can be made so that the expression on the right-hand side for a system of x 's, which belongs to the boundary, does not become arbitrarily small, but always remains greater than a certain lower limit (see the preceding article).

The expression, however, on the interior of the realm of the x 's may be arbitrarily small, viz.: when $x_1 = x_2 = \dots = x_n = 0$.

For this system of values the left-hand side of [8] is equal to

$$\sum_{\mu=1}^{\mu=n} y_{\mu}^2.$$

We have therefore found the following result: *We can give limits g to the variables x , and to the y 's the limits h in such a way that the expression [7] for systems of values of the x 's, which belong to the boundary of the realm, is always greater than it is for the zero position ($x_1 = x_2 = \dots = x_n = 0$).*

Hence the position for which the expression [7] is a minimum must necessarily lie within the realm of the x 's; and we may be certain that within the realm of the x 's there is a position where [7] has its smallest value.

31. In order to find the minimal position of [7] which was shown to exist in the previous article, we must, following the rules laid down in Chapt. III., differentiate the function [7] and put the first partial derivatives = 0.

This gives

$$[9] \quad \sum_{\nu=1}^{\nu=n} (F_{\nu} - y_{\nu}) \frac{\partial F_{\nu}}{\partial x_{\mu}} = 0.$$

($\mu = 1, 2, \dots, n$)

These n equations can, in case the determinant

$$[10] \quad \left| \frac{\partial F_\nu}{\partial x_\mu} \right|$$

$$(\nu, \mu = 1, 2, \dots, n)$$

is different from zero, only exist, if the quantities within the brackets vanish.

[10] is identical with the determinant [4], and (see art. 28) it may be always brought about through suitable choice of the limits g of the x 's that [4] is different from zero, if only the determinant [2], as by hypothesis is the case, is different from zero.

Hence the equation [9] can only be satisfied if

$$[1a] \text{ or } [5] \quad y_\nu = F_\nu(x_1, x_2, \dots, x_n).$$

We have therefore found that, *since there is certainly a system of values of the x 's for which the function $\sum_{\nu=1}^{\nu=n} (y_\nu - F_\nu)^2$ is a minimum, there must also be a system within the realm of the x 's for which the equations [1a] are satisfied, if to the y 's definite values in their realm are arbitrarily given.*

32. We must further see whether within the fixed realm there is one or several systems of values of the x 's, that satisfy the equations [1a] with prescribed values of the y 's which lie within definite limits.

In order to establish this, we assume that $(x'_1, x'_2, \dots, x'_n)$ is a second system of values that satisfy the equations [1a]; we must then have the equations

$$[11] \quad F_\nu(x'_1, x'_2, \dots, x'_n) - F_\nu(x_1, x_2, \dots, x_n) = 0.$$

$$(\nu = 1, 2, \dots, n)$$

Developing by Taylor's theorem, we have, when we consider only terms of the first dimension

$$[11a] \quad \sum_{\rho=1}^{\rho=n} (x'_\rho - x_\rho) \left\{ \frac{\partial F_\nu}{\partial x_\rho} + \overline{X}_{\nu, \rho} \right\} = 0.$$

$$(\nu = 1, 2, \dots, n)$$

The $\overline{X}_{\nu, \rho}$ are functions which depend upon the x' 's and x 's and vanish with these quantities.

We will determine the n unknown quantities $x'_1 - x_1, x'_2 - x_2, \dots, x'_n - x_n$ from the n linear equations [11a].

For small values of x and x' the determinant

$$[12] \quad \left| \frac{\partial F_\nu}{\partial x_\rho} + \overline{X}_{\rho, \nu} \right|$$

$$(\nu, \rho = 1, 2, \dots, n)$$

will be little different from the determinant [10].

We may therefore make the limits g of the x 's so small that [12] is different from zero for all the x 's and x' 's which belong to the realm.

But then the equations [11a] are only satisfied for

$$x'_\rho = x_\rho \quad (\rho = 1, 2, \dots, n);$$

that is, *there exists within the realm in question no second system of the x 's which satisfies the equations [1a].*

We have therefore come to the following result:

It is possible so to determine the limits g and h that with every arbitrary system of the y 's in which each single variable does not exceed its definite limiting value, the given equations [1a] are satisfied by ONE and ONLY ONE system of the x 's, in which these quantities likewise do not exceed their limits.

The first part of the theorem given in art. 26 is thus proved.

REMARK.—We have assumed that we have to do only with real quantities. The discussion, however, is not restricted to such quantities, as it is easy to prove that the same developments may be also made for complex variables.

33. The values of the x 's which were had from the equations

$$[1a] \quad y_\nu = \sum_{\rho=1}^{\rho=n} \left(a_{\nu\rho} + \overline{X}_{\nu\rho} \right) x_\rho$$

$$(\nu = 1, 2, \dots, n)$$

may be derived in the manner given in art. 26:

If we write

$$\left| a_{\nu\rho} + \sum_{\nu} X_{\nu\rho} \right| = A',$$

$$(\nu, \rho=1, 2, \dots, n)$$

the linear solution of the equations [1a] is

$$[3b] \quad x_{\rho} = \sum_{\nu=1}^{\nu=n} \frac{A'_{\nu\rho}}{A'} y_{\nu},$$

$$(\rho=1, 2, \dots, n)$$

where $A'_{\nu\rho}$ denotes the corresponding subminor of A' . Now A' is a definite quantity which lies within certain finite limits; the same is also true of $\frac{1}{A'}$. $A'_{\nu\rho}$ is found in a similar manner. Hence the quantities $\frac{A'_{\nu\rho}}{A'}$ are finite quantities which lie between definite limits; and therefore, if the y 's become infinitely small, the x 's will also become infinitely small; that is, those systems of values of the x 's, which satisfy the equations [1] under the named conditions, are—as has also been shown in art. 28—so formed that they become infinitely small with the y 's.

We can now show that the x 's are continuous functions of the y 's.

Let $(b_1, b_2, \dots, b_n)$ be a definite system of values of the y 's and let the system $(a_1, a_2, \dots, a_n)$ of the x 's correspond to this system of the y 's.

If we then write

$$[13] \quad \left\{ \begin{array}{l} y_{\lambda} = b_{\lambda} + \eta_{\lambda} \\ x_{\lambda} = a_{\lambda} + \xi_{\lambda} \end{array} \right\},$$

$$(\lambda=1, 2, \dots, n)$$

the system of equations [1a] or [1] goes into

$$\eta_{\lambda} + b_{\lambda} = F_{\lambda}(a_1 + \xi_1, a_2 + \xi_2, \dots, a_n + \xi_n),$$

$$(\lambda=1, 2, \dots, n)$$

or

$$[1b] \quad \eta_\lambda = F_\lambda (a_1 + \xi_1, a_2 + \xi_2, \dots, a_n + \xi_n) - F_\lambda (a_1, a_2, \dots, a_n). \\ (\lambda = 1, 2, \dots, n)$$

Developing this expression according to powers of the ξ 's, we have

$$[1c] \quad \eta_\lambda = \sum_{\mu=1}^{\mu=n} A'_{\lambda\mu} \xi_\mu, \\ (\lambda = 1, 2, \dots, n)$$

where the $A'_{\lambda\mu}$ are functions of the a 's and ξ 's. If now the ξ 's are infinitely small, we may limit the $A'_{\lambda\mu}$ to the first derivatives of F_λ . In this case we denote the coefficients of [1c] by $A_{\lambda\mu}$, so that

$$A_{\lambda\mu} = \frac{\partial F_\lambda}{\partial x_\mu} \text{ for } (x_1 = a_1, x_2 = a_2, \dots, x_n = a_n), \\ (\lambda, \mu = 1, 2, \dots, n)$$

and the determinant of the equations [1] goes into

$$\left| \frac{\partial F_\lambda}{\partial x_\mu} \right| \text{ for } (x_1 = a_1, x_2 = a_2, \dots, x_n = a_n). \\ (\lambda, \mu = 1, 2, \dots, n)$$

If the x 's lie within definite limits, this determinant remains always above a definite limit. We may therefore say that the determinant has a value different from zero. Consequently the condition that the equations [1c] may be solved, is satisfied, and it is seen that infinitely small values of the ξ 's must correspond to infinitely small values of the η 's.

This means nothing more than that the functions of the x 's are continuous functions of the y 's.

34. Our investigations are true under the assumption that the functions F_λ are continuous, that their first derivatives exist and likewise are continuous within certain limits. We need know absolutely nothing about the second derivatives.

Of the x 's, of which we already know that they exist as functions of the y 's and vary in a continuous manner with them, we may now likewise prove that they, considered as functions of the y 's, have derivatives which are continuous functions of the y 's.

We have then proved that the x 's are such functions of the y 's as the y 's are of the x 's.

The proof in question may be derived from the following considerations:—If from [1c] we express the ξ 's in terms of the η 's, we have

$$[3c] \quad \xi_{\mu} = \sum_{\lambda=1}^{\lambda=n} \frac{\Delta_{\lambda,\mu}}{\Delta} \eta_{\lambda}.$$

($\mu=1, 2, \dots n$)

The $\frac{\Delta_{\lambda,\mu}}{\Delta}$ are continuous functions of the ξ 's, and the ξ 's are continuous functions of the η 's. Hence $\frac{\Delta_{\lambda,\mu}}{\Delta}$ may be represented as continuous functions the η 's.

If the η 's become infinitely small, then the ξ 's become infinitely small, and we have limits for $\frac{\Delta_{\lambda,\mu}}{\Delta}$.

If we have *in general* a function of the n variables $x_1, x_2, \dots x_n$, and if we consider the difference

$$f(a_1 + \xi_1, a_2 + \xi_2, \dots a_n + \xi_n) - f(a_1, a_2, \dots a_n),$$

it is seen that it may be written in the form

$$\sum_{\lambda=1}^{\lambda=n} (b_{\lambda} + H_{\lambda}) \xi_{\lambda},$$

where the H_{λ} depend upon the ξ 's and become infinitely small with these quantities, and the b_{λ} are the partial differential quotients of f with respect to x_{λ} for the system of values $(a_1, a_2, \dots a_n)$. From this it results the x 's are not only continuous functions of the y 's, but also that the derivatives of the first order of the functions exist.

We have indeed the derivatives of the first order, if in the expressions $\frac{\Delta_{\lambda,\mu}}{\Delta}$ we write the ξ 's equal to zero.

The quantities $\frac{\Delta_{\lambda,\mu}}{\Delta}$, however, become then in accordance with

[1c], the quantities which we would have in [1c], if we had at first written $A_{\lambda\mu}$ instead of $A'_{\lambda\mu}$.

But the quantities $A_{\lambda\mu}$ are continuous functions of $a_1, a_2, \dots, a_n$. We may therefore say that the differential quotients $\frac{\Delta_{\lambda\mu}}{\Delta}$ are continuous functions of the variables x . •

35. For the complete solution of the second part of the theorem in art. 26, we have yet to show that the expressions [3b] may be reduced to the form [3a]:

For this purpose we must bring the quantities $\frac{A'_{\lambda\mu}}{A'}$ in [3b] (art. 32) to the form

$$\frac{A'_{\lambda\mu}}{A'} = b_{\lambda\mu} + Y_{\lambda\mu},$$

where $b_{\lambda\mu}$ is the value of $\frac{A'_{\lambda\mu}}{A'}$ when all the x 's are equal to zero. $Y_{\lambda\mu}$ is a function of the x 's, but the x 's are functions of the y 's, so that $Y_{\lambda\mu}$ is a function of the y 's which vanishes when they vanish.

We may therefore in reality write [3b] in the form [3a]

$$[3a] \quad x_\mu = \sum_{\lambda=1}^{\lambda=n} \left(b_{\lambda\mu} + Y_{\lambda\mu} \right) y_\mu. \\ (\mu=1, 2, \dots, n)$$

36. There may arise cases in which we know nothing further of the functions F_λ —as was assumed in art. 26—than that they are real continuous functions.

We cannot then conclude, for example, that the x 's may be developed in powers of the y 's; but we may reduce the equations to the form [3c] and show that the equations [1a] are solvable.

The theorem which has been proved is of great importance

when applied to special cases, even for elementary investigations.

If, for example, the equation $f(x, y)=0$ is given, then it is taught in the differential calculus how we can find the derivative of y considered as a function of x .

If we assume that the variables x and y are limited to a special realm, where the two derivatives with respect to x and y do not vanish, and therefore the curve $f(x, y)=0$ has no singular points; and, if the equation is satisfied by the system (x_0, y_0) , we may write $x=x_0+\xi$, $y=y_0+\eta$. We have then $f(x_0+\xi, y_0+\eta)=0$, and we may prove with the aid of the theorem in art. 27 that η is a continuous function of ξ and has a first derivative. Not before this has been done, have we a right to differentiate and proceed according to the ordinary rules of the differential calculus.

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